Economic Growth when Knowledge is Concentrated*

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Abstract

Firms' innovation outcomes depend on their ability to attract and retain talented inventors. What market frictions prevent the sorting between firms with high innovation potential and high-productivity inventors? How does this sorting impact aggregate innovation, growth and welfare? We address these questions both empirically and theoretically. Empirically, we show that firms facing strong competition in the product market employ more productive inventors, while less productive inventors tend to be allocated in concentrated industries. Theoretically, we embed a frictional labor market for inventors into an endogenous-growth model of strategic innovation. In line with the data, the model predicts that high-productivity inventors are disproportionately employed in firms that operate in competitive industries. We then use the model to quantify the growth and welfare implications of this inventor sorting. Our results show that matching frictions in the market for inventors impede the allocation of high-productivity inventors to firms with high implementation intensity, and are responsible for a 32% loss in economic growth. Industrial policies that subsidize R&D spending relax these frictions by boosting inventor productivity, helping high-quality inventors reallocate to firms with high implementation incentives. Under optimal subsidies, growth increases as much as 74 basis points, closing most of the gap in missing growth caused by frictions in the market for inventors.

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JEL Classification: L16, J6, O3, O4.

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1 Introduction

In idea-based growth models, economic growth arises from people generating new ideas. Generating a new idea, however, does not enhance productivity by itself. Ideas translate into productivity growth as a result of the implementation and commercialization efforts of firms who, driven by profit motives, transform ideas into new products and processes. Thus, long-run productivity growth depends on the efficiency with which ideas are integrated into production, and particularly on the market forces that determine the matching between inventors (those that create ideas) and firms (those that implement them). From this perspective, aggregate innovation and growth are highest when inventors with the best ideas are matched with firms with the highest innovation potential and R&D productivity, where the social contribution of ideas is the greatest. The efficient functioning of the market for inventors is, therefore, essential for achieving a socially optimal allocation.

In this paper, we study the labor market for inventors to understand the sorting patterns of inventors into firms and to quantify their aggregate implications. Empirically, we present new findings on inventors' mobility and their allocation across firms. Theoretically, we develop an endogenous-growth model featuring a frictional market for inventors, where firms competing in the product market strive to match with the best inventors. Efficiency in the market for inventors determines the extent to which firms with high innovation capabilities are able to hire and retain the most talented inventors. Our model allows us to quantify the growth and welfare effects of matching frictions that prevent the efficient allocation of inventors across firms.

We begin by documenting novel empirical facts about the labor market for inventors. By combining patent data from the US Patent and Trademark Office (USPTO) with firms' financial information from Compustat North America, we identify which inventors work for which firms and relate inventor characteristics to balance sheet information about their employers. We document that there is significant turnover in the market for inventors: on average, 6.6% of inventors in the United States change employer every year, with mobility rates being higher for inventors that have better past innovation outcomes. We find that firms that operate in competitive sectors have higher hiring rates for highly innovative inventors compared to firms in concentrated industries. As a result, high-productivity inventors are relatively more likely than low-productivity inventors to work in competitive industries.

Motivated by these findings, we build an innovation-based growth model with a frictional labor market for inventors. We merge a model of competitive innovation with a firm-dynamics model of directed search. A standard feature of innovation-based growth models is that firms can hire R&D inputs on frictionless markets. In our model, in contrast, firms must access a frictional market for inventors before they can innovate to improve on their productivity. As a result of their research efforts, inventors are heterogeneous in the quality of their ideas, and firms compete to attract the best inventors and implement their ideas.

The model economy consists of many sectors. In each sector, two large firms produce imperfectly substitutable varieties of the same good and compete in prices à la Bertrand for market leadership. A competitive fringe of small firms serves the remainder of the market. Large firms are heterogeneous in productivity, which evolves endogenously through the successive implementation of ideas. Firms that employ inventors of higher ability can achieve greater productivity advancements by implementing their inventor's ideas. The matching between inventors and firms occurs in a frictional labor market. Search in this market is directed on both the inventor and the firm side: firms announce wage contracts in order to attract unattached inventors, who optimally direct their search based on these offers. Meeting rates are therefore determined by the optimal choices of inventors and firms, so the model generates endogenously heterogeneous job-filling rates across firms and industries.

In equilibrium, firms' innovation incentives are driven by competition in the product market: market leaders try to innovate to widen their lead, while followers try to catch up with the leader. Intense competition in the product market induces more aggressive innovation investments by firms, while innovation efforts are lower in sectors where the productivity gap between leader and follower is larger. As a consequence, firms in more competitive industries offer better wages to more creative inventors, and in equilibrium highly productive inventors tend to be allocated in such industries. Our framework thus allows for heterogeneity on both *R&D intensity* and *R&D productivity*, linking them together: it is not only the aggregate arrival rate of innovations that matters for economic growth, but also the distribution of match qualities of inventors to firms. In particular, economic growth will be highest when inventors with the best ideas work for the firms that have the highest incentives to implement them.

We calibrate the model to our sample of micro-data that merges patents with firms' balance sheet information, and we validate our calibration strategy by showing that the model replicates the allocation of inventors across firms and the patterns of inventor mobility that we observe in the data. In the model, as in the data, many firms that are far from their competitors employ low-productivity inventors, while firms that are close to their competitors tend to employ high-productivity inventors. There are, however, many high-productivity inventors that are inefficiently employed by dominant firms, who have little incentive to implement innovations. This happens because of the strategic nature of firms' competition for inventors: dominant firms employ talented inventors to discourage innovation by their competitors and defend their market leadership.

In the last part of the paper, we use our calibrated model to study the growth and welfare implications of inventor-firm sorting. To quantify the effects of frictions in the allocation of inventors to firms, we first show that increasing matching efficiency in the market for inventors (which induces an increase in inventor labor market mobility) favors the reallocation of high-productivity inventors to industries in which firms have high implementation intensity. As misallocation in the market for inventors declines and firms' implementation spending increases, firms compensate their inventors better, creating stronger incentives among inventors to strive to find more disruptive idea. Interestingly, the rate of new idea creation (the probability of generating an innovation for the average inventor) declines, but as realized ideas are now more radical, they are also much more likely to be implemented by firms. As a result, even though there are fewer new ideas in equilibrium, economic growth soars. Compared to a counterfactual economy with no matching frictions and full mobility among inventors, the rate of economic growth in the calibrated economy is almost a full percentage point lower. Or, in other words, frictions in the labor market for inventors are responsible, through the lens of our model, for a 32% loss in economic growth.

Finally, we uncover a new allocative role in industrial policy: classic R&D subsidies (here, to inventor's research spending or to firm's implementation spending) foster inventor mobility, trigger a reallocation of talent across and within industries, and alleviate efficiency losses coming from the mismatch in incentives between inventors with good ideas and firms who seek to implement them. The optimal subsidy to the research expenditures incurred by inventors increases economic growth by 0.12 percentage points, and leads to a 2.90% increase in consumption-equivalent welfare for the representative consumer. The optimal subsidy to the implementation expenditures of firms is even more impactful, leading to a large welfare gain of 22.5%. In particular, this policy increases growth by 0.74 percentage points, i.e. it closes about fourth-fifths of the gap in growth caused by the sorting frictions in the economy.

When we study the distributional implications of each policy, we find stark differences. Subsidies to firms' implementation costs increase economic growth not because they induce firms to hire more talented inventors, but because they lead them to implement their inventors' ideas more often. That is, growth goes up because more resources are used, albeit less efficiently. In contrast, subsidies to inventors' research costs increase economic growth not because their ideas get implemented more often, but because they induce inventors to accumulate more human capital and to produce more radical ideas. That is, growth goes up because resources are used more efficiently, albeit fewer of them are used.

Related literature This paper builds on and contributes to several strands of the literature. Theoretically, our model builds on the branch of the endogenous-growth literature that incorporates product market competition and strategic interaction between firms in general equilibrium models of economic growth (Aghion, Harris, Howitt and Vickers (2001), Aghion, Bloom, Blundell, Griffith and Howitt (2005), Acemoglu and Akcigit (2012), Akcigit and Ates (2023)). We contribute to this literature by embedding a frictional labor market for inventors. This allows us to study the growth and welfare consequences of the misallocation between inventors, who generate new ideas, and firms, who implement these ideas —a dimension that remains unexplored in this literature. Our modeling of the market for inventors, in turn, borrows from the dynamic directed search literature with heterogeneous firms (e.g. Kaas and Kircher (2015) and Schaal (2017)). Thus, we also contribute to the literature on frictional search by applying it to the market for inventors and by studying the implications for aggregate productivity growth.[1]

Conceptually, our paper contributes to the study of the inefficiencies involved in the processes through which new ideas are generated and embedded in economic activity. Concerning the generation of ideas, a recent literature studies selection into the pool of inventors and misallocation effects in "who becomes an inventor" (see e.g., Aghion, Akcigit, Hyytinen and Toivanen (2017), Celik (2023b), Akcigit, Pearce and Prato (2025b) and Akcigit, Alp, Pearce and Prato (2025a)). We share with this literature the interest in the growth and welfare implications of the (mis-)allocation of talent, but our focus is on the allocation of inventors across firms, rather than on the supply of inventors per se. Hence, we stress the inefficiencies related to the commercialization of new ideas. Other papers have focused on the role of financial frictions in reducing or delaying the incorporation of new ideas into production (e.g. Chiu, Meh and Wright (2017) and Celik (2023a)), while our attention is on frictions in the labor market. By arguing that inventors (ideas) might be misallocated across firms, our paper is also related to Akcigit, Celik and Greenwood (2016), which studies inefficiencies in the ownership of ideas. In a similar vein to Akcigit *et al.* (2016), we argue that an efficient functioning of the market through which inventors

[1]To the best of our knowledge, the only other paper that does this is Babalievsky (2023). As we do, Babalievsky (2023) proposes a growth model that features a frictional labor market for inventors to study the consequences for growth of the allocation of inventors across firms. However, in his framework, search is random, so that meeting rates between inventors and firms are exogenous. By contrast, in our model, search is directed and meeting rates are determined endogenously by the values that firms promise to their inventors. Our model can therefore rationalize the novel micro evidence on inventors' mobility and firms' hiring policies across industries that we document. Moreover, it allows for a quantification of the macroeconomic impact of this micro behavior, endogenizing firms' and inventors' responses in the labor market to changes in policy.

can be reallocated across firms can generate large growth and welfare effects.

By studying the matching between inventors and firms, our paper is closely related to the recent literature, initiated by Bloom, Jones, Van Reenen and Webb (2020), that investigates the determinants of aggregate R&D productivity. Within this literature, some authors have pointed to misallocation of R&D resources across firms as a driver of the recent productivity slowdown in the US economy. Lehr (2022) and Ayerst (2022) provide evidence of declining allocative efficiency in the R&D sector; Manera (2022) shows that dominant firms in concentrated sectors have attracted a disproportionate share of inventors; Akcigit and Goldschlag (2023) document that inventors are increasingly concentrated in old, large incumbents and less likely to work for young firms; and Fernández-Villaverde, Yu and Zanetti (2025) show that defensive hiring of researchers may explain low TFP growth in spite of high R&D spending.[2] While the focus of these papers is on the changing patterns of firm-inventor sorting over time in the data, we still understand little about the determinants of the sorting of inventors into firms in the cross-section and its quantitative implications for innovative capacity and growth. Our paper tries to fill this gap.[3]

Outline The remainder of the paper is structured as follows. Section 2 describes the data and presents the results of our empirical analysis. Section 3 introduces the theoretical model and characterizes its equilibrium. Section 4 presents our calibration strategy and discusses some of the model results, showing that the model's predictions align well with the empirical evidence. Section 5 presents the results of our quantitative policy exercises, which assess the implications of the market for inventors. Section 6 concludes.

2 Empirical Analysis

2.1 Data Sources

To study the labor market dynamics of inventors and their employers, we combine data from two main sources: (i) patent data from the PatentsView's database of the United States Patent and Trademark Office (USPTO); and (ii) balance-sheet and income statements information for listed firms in the United States from Compustat North America.

[2] Firms engaging in defensive hiring to reduce the probability that competitors will innovate is a force that is also present in our model.

[3] Like us, Baslandze and Vardishvili (2026) use patent data to trace inventor mobility across firms. However, while the focus of our paper is to study the effect of sorting between inventors and firms, they study how inventor mobility can result in new highly innovative spinouts of older firms.

PatentsView The PatentsView database provides comprehensive information on all patents granted by the USPTO since 1976. We leverage data on utility patents granted up to 2018, which allows us at least 5 years to observe forward citations to the latest patents. Our dataset contains 6,203,450 patents granted between 1976 and 2018, and 3,180,835 unique inventors. We assign patents to specific years according to their filing date to the USPTO, which ensures that the patent is assigned to the year closest to the actual innovation. For each patent, the dataset includes information about both the inventors who contributed to the innovation and on its assignees. This allows us to identify which inventors work for which firms and characterize the empirical patterns of inventors' mobility across firms.

Compustat We rely on Compustat North American Fundamentals for financial statement information about U.S.-listed firms. We map patent assignees from PatentsView to firms in Compustat by using the crosswalk developed by Dyèvre and Seager (2023). Doing so allows us to relate inventors' characteristics to balance-sheet information about their employers. The crosswalk includes 2,564,512 patents, comprising 41% of all patents granted by the USPTO from 1976 to 2018.[4] This sample of patents are assigned to 8,182 unique Compustat firms, and 1,127,381 unique inventors are attached to these firms.

2.2 Variable Construction

2.2.1 Inventors' productivity

The main inventor characteristic that we are interested in is a measure of their innovative output. As is standard in the literature (e.g. Hall, Jaffe and Trajtenberg, 2005), we measure the quality $q(p)$ of a patent p by the number of forward citations it receives over a 5-year window from its grant date, $g(p)$:

$$q(p) = \sum_{t=g(p)}^{g(p)+4} \#\text{citations}_t. \tag{1}$$

We define the flow productivity of inventor j in year t, denoted q_{jt}, as the total quality of the set of (granted) patents $\mathcal{P}_{jt}$ that she applied for in that year, $q_{jt} = \sum_{p \in \mathcal{P}_{jt}} q(p)$.[5] Given this flow productivity measure, we construct the stock, or cumulative, productivity of

[4] Patents assigned to Compustat firms account for 48% of total forward citations received by patents granted by the USPTO.

[5] For robustness, we also consider alternative measures of inventors' productivity based on different definitions of patent quality: forward citations in a 3-year window from the grant date; forward citations excluding self-citations; and citations per author. See Appendix B for further details.

inventor j, denoted Q_{jt}, as her citations-weighted patent stock at the start of year t:

$$Q_{jt} = \frac{1}{t - t_0(j)} \sum_{s=t_0(j)}^{t-1} q_{js}, \tag{2}$$

defined for all $t \geq t_0(j) + 1$, where $t_0(j)$ is the year in which inventor j applied for its first patent. We use Q_{jt} as our benchmark measure of an individual inventor's productivity. This measure takes into account both the quantity and the quality of an inventor's past innovations, and it is normalized by the inventor's tenure in the patent market to adjust for the fact that older inventors are expected to have higher cumulative productivity due to having had more years to accumulate citations.[6]

Table C.1 in the Appendix reports descriptive statistics for this and several other measures of inventors' innovative output, both conditioning on patenting years (Panel A) and over their entire career (Panel B), i.e. including unproductive years of inventors.

2.2.2 Inventors' employment histories

As we do not have access to actual employment information for inventors, we identify their affiliation over time from the information contained in the patents. We say that inventor j is employed by firm i in year t if the majority of the patents that j applied for in that year are assigned to firm i. Then, based on the application dates and assignees of their patents, we create each inventor's employment history.

As in Akcigit, Caicedo, Miguelez, Stantcheva and Sterzi (2018), we assume that inventor j is employed in firm i from the first year the majority of her patents are assigned to firm i to either (i) the year before the majority of her patents are assigned to firm k, or (ii) the last year the majority of her patents are assigned to firm i, in case she does not have any further patent afterwards. We assign zero patents to the “unproductive" years of inventors, instead of treating them as missing. Based on inventors' employment histories, we identify inventors who move across firms. We then compute hiring and separation rates of inventors at the firm level, defined as the number of inventors who move to or leave firm i in year t as a fraction of the total number of inventors in firm i in year $t - 1$.

[6] It could be argued that q_{jt} and Q_{jt} reflect not only inventor's productivity but also the innovation capabilities of the firm that the inventor works for. As shown in Appendix B.4, using the Abowd, Kramarz and Margolis (1999) (AKM) method, we find that inventor fixed effects account for 85% of explained variance in q_{jt} vs. 8% for firm fixed effects. This lends support to using q_{jt} and Q_{jt} as measures of inventor productivity.

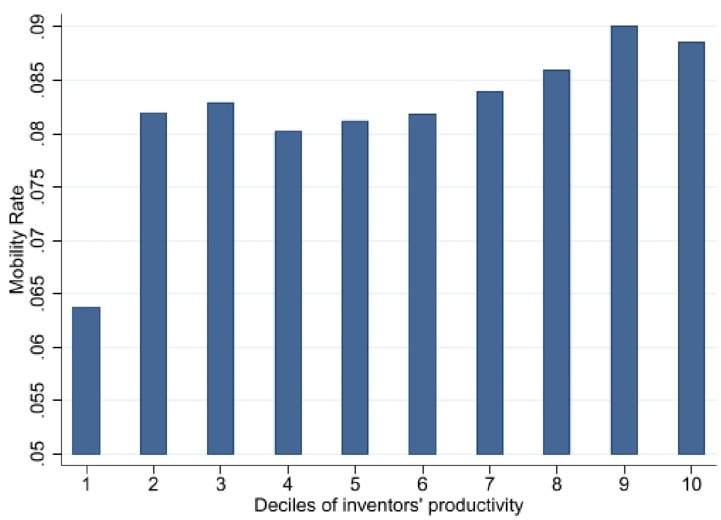
2.3 Empirical Findings

Using these variables, we document new empirical facts about the market for inventors. We characterize the empirical patterns of inventors' mobility and the distribution of inventors across firms. These will serve to motivate our model's assumptions and test its predictions.

2.3.1 Inventors' mobility

We begin by establishing salient cross-sectional features of the market for inventors in the U.S. economy. First, in the full USPTO sample, on average 6.6% of inventors change employer every year. This share has been increasing over time, from 4-5% in the 1980s to 9-10% in the 2010s. In the full USPTO sample, the average number of years between an inventor's first and last patent is 6.7. Inventors have on average 5 patents across 1.5 firms over their career. This suggests a non-negligible amount of turnover: a move occurs once every 4.6 years on average.

Figure 1: Mobility rates by inventors' productivity deciles



Interestingly, mobility rates are higher among inventors with better past innovation outcomes, as measured by Q_{jt} (defined in equation (2)). To describe the patterns of inventor mobility by inventor productivity, we group inventors into deciles based on the distribution of Q_{jt}, in year t. We compute mobility rates year-by-year for each decile and then average across years. Figure 1 shows the results, showing that mobility rates tend to be higher among inventors with better past innovation outcomes.

2.3.2 Firms' hiring policies

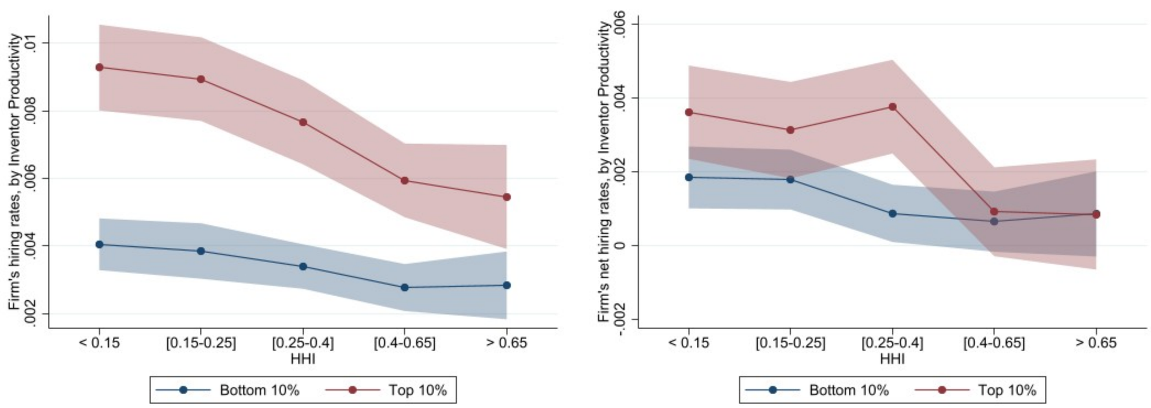
Next, we characterize empirically the behavior of firms in the market for inventors. In particular, we study how the intensity of competition that firms face in the product market affect their hiring rates of inventors of different past productivity.

To do so, we estimate the following regression at the firm level:

$$Y_{it} = \beta_0 + \sum_{k=1}^{5} \beta_k D_k^{HHI_{s(i)t}} + X_{it}'\gamma + Z_{s(i)t}'\delta + \tau_t + u_{it}. \tag{3}$$

The outcome variables that we consider are the firm's hiring rate of inventors and its net hiring rate, defined as the difference between hires and separations. $\{D_k^{HHI_{s(i)t}}\}_{k=1}^{5}$ are group dummies based on the HHI index of the 6-digit NAICS industry where firm i operates; X_{it} includes controls for the firm's age, employment, R&D stock, profitability, leverage and market-to-book ratio; $Z_{s(i)t}$ controls for the number of firms in the industry and industry sales; and τ_t are year fixed effects. We run this regression separately for (gross and net) hiring rates of inventors with different past innovation outcomes.

Figure 2: Hiring and net hiring rates of inventors



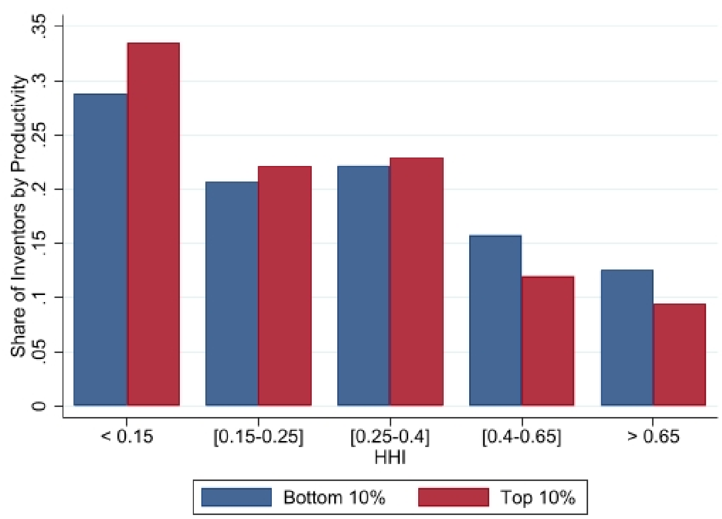
Notes: Hiring (left-hand panel) and net hiring (right-hand panel) rates of inventors. Fitted values from regression (3). Controls: firm's age, employment, R&D stock, profitability, leverage and market-to-book ratio; number of firms in the industry, industry sales; year fixed effects. Standard errors are clustered at firm and year level. Numbers are scaled using the mean of the omitted group (HHI < 0.15), keeping controls at their means.

The estimation results are visualized in Figure 2 for hiring rates of inventors in the top (in red) and in the bottom (in blue) deciles of the inventor's productivity distribution in a given year. We find that firms that operate in more competitive sectors have higher hiring rates of inventors compared to firms that operate in more concentrated industries. The relationship is however relatively flat for hiring rates of inventors with low past productivity, and much more pronounced for inventors with good past innovation outcomes. These patterns hold for both gross and net hiring rates.[7]

2.3.3 Distribution of inventors across industries

As a final piece of evidence, in Figure 3 we look at the empirical distribution of inventors across different 6-digit NAICS industries, grouped by HHI index. The blue bars refer to the distribution of inventors in the bottom decile of inventor productivity distribution in a given year, while the red bars refer to inventors in the top decile. We find that low-productivity inventors are more likely than high-productivity inventors to work in concentrated sectors (as measured by the industry's HHI), while high-productivity inventors are more likely than low-productivity inventors to work in competitive sectors. There is however a non-negligible share of high-productivity inventors that are employed in concentrated sectors.

Figure 3: Distribution of inventors across industries



Notes: Bars represent shares of inventors of a given productivity decile in a given year (bottom 10%, blue; top 10% red) employed in different 6-digit NAICS industries, grouped by HHI index. Inventor's productivity is measured by Q_{jt}, as explained above.

As we will show below, our model is able to replicate this allocation of inventors to firms across industries, and shows that this allocation might be quantitatively important for growth and welfare.

[7] In Appendix B we also document that, within industries, there is an hump-shaped relationship between hiring rates of inventors and firms' relative sales. This is reminiscent of various previous studies that find similar hump shapes between innovation outcomes and competition faced by the firm.

3 Model

3.1 Environment

Time is continuous, runs forever, and is indexed by $t \in \mathbb{R}_+$. The economy is populated by three types of agents: a representative household, firms, and inventors. All agents are risk-neutral and infinitely-lived, and share a common time discount rate, $\rho > 0$.

Preferences and technology The representative household maximizes lifetime utility:

$$\int_0^{+\infty} e^{-\rho t} \ln(\boldsymbol{C}_t) dt, \tag{4}$$

where $\boldsymbol{C}_t > 0$ is the consumption of a single final good, which is the numeraire and whose price we normalize to one. Every instant of time, the household inelastically supplies $\boldsymbol{L} = 1$ units of so-called *production labor* at wage w_t^P, earned in a perfectly competitive labor market. The household owns all the firms in the economy and accumulates assets $\boldsymbol{A}_t$. The budget constraint satisfies $\partial_t \boldsymbol{A}_t \equiv \frac{d\boldsymbol{A}_t}{dt} = r_t \boldsymbol{A}_t + w_t^P - \boldsymbol{C}_t$, where r_t is the interest rate of return on corporate asset holdings.

The final good, denoted $\boldsymbol{Y}_t$, is produced competitively combining intermediate goods Y_{jt} from a measure-one continuum of industries $j \in [0, 1]$ using a Cobb-Douglas technology:

$$\boldsymbol{Y}_t = \exp\left(\int_0^1 \ln(Y_{jt}) dj\right). \tag{5}$$

In turn, industry j's output, Y_{jt}, is a constant-elasticity-of-substitution (CES) composite of the varieties of intermediate good j that are produced by two active *large* firms in the industry, called i and $-i$, producing outputs y_{ijt} and y_{-ijt}, respectively, and by a set of *small* firms of unit mass (collectively, a "competitive fringe"), collectively called c, that produce homogeneous products. Industry j's output is given by

$$Y_{jt} = \left(\sum_{f=i,-i,c} y_{fjt}^{\sigma}\right)^{\frac{1}{\sigma}}, \tag{6}$$

where $\sigma \in (0, 1)$, so that the elasticity of substitution between the three varieties of good j is equal to $\varepsilon \equiv \frac{1}{1-\sigma} > 1$. As each small firm $k \in [0, 1]$ in the competitive fringe produces a homogeneous product, so $y_{cjt} = \int_0^1 y_{kcjt} dk$. All variety producers use the same linear technology in production labor:

$$y_{fjt} = q_{fjt} l_{fjt}, \ \forall f \in \{i, -i\}, \quad \text{and} \quad y_{kcjt} = q_{cjt} l_{kcjt}, \ \forall k \in [0,1] \tag{7}$$

where q_{fjt} is labor productivity of large firm $f \in \{i, -i\}$ at time t, and q_{cjt} is the productivity of any small firm k in the fringe (i.e., each small firm operates with the same productivity within an industry). Since each small firm is infinitesimal and produces a homogeneous product, the fringe behaves as a price-taking representative firm. However, we assume that large firms behave strategically, competing in prices à la Bertrand.

Innovation In the tradition of step-by-step innovation models, we assume that large firms can advance their productivity on a ladder via investments to realize innovations. Different from the traditional models, however, the process of finding new ideas is treated separately from the process of implementing them. While large firms can implement ideas to realize innovations, they need to employ an inventor who has first generated them.

Inventors are heterogeneous in their *knowledge capital*, which determines the extent by which a large firm's productivity q_{fjt} will increase if the firm successfully implements the inventor's idea.[8] Knowledge capital evolves over time as a result of the inventor's decisions (as described below), it is fully transferable across industries and firms, and takes values in a discrete and finite set, $\kappa_t \in \mathbb{K} \equiv \{\underline{\kappa}, \underline{\kappa}+1, \ldots, \overline{\kappa}-1, \overline{\kappa}\}$, where $\underline{\kappa} \geq 1$ and $\overline{\kappa} > \underline{\kappa}$ are positive integers. To implement an innovation at Poisson arrival rate x_{fjt}, large firm $f \in \{i, -i\}$ must pay $\xi x_{fjt}^{\phi} \boldsymbol{Y}$ units of the final good, where $\xi > 0$ and $\phi > 1$. If a type-κ inventor's idea is implemented at time t, the firm's productivity advances by κ steps on the quality ladder, so that $q_{fj,t+\Delta t} = \lambda^{\kappa} q_{fjt} + o(\Delta t)$ with probability $x_{fjt}\Delta t$, where $\lambda > 1$.

Knowledge capital does not immediately diffuse across firms. Thus, firms can wedge a productivity gap with respect to their competitors by implementing their inventors' ideas. Employing better inventors allows large firms to take larger steps relative to their competitors. Denoting by $n_{ijt} \in \{-\overline{n}, \ldots, -1, 0, 1, \ldots, \overline{n}\}$ the technology gap between large firms i and $-i$ in industry j, where $\overline{n} > 0$ is a large integer, we have:

$$\frac{q_{ijt}}{q_{-ijt}} = \lambda^{n_{ijt}}. \tag{8}$$

If $n_{ijt} > 0$, firm i is said to be the *market leader* in industry j; if $n_{ijt} < 0$, firm i is the *market follower* in industry j; and if $n_{ijt} = 0$, firms i and $-i$ are said to be *neck-to-neck*. We assume that market followers benefit from knowledge diffusion at an exogenous Poisson arrival

[8] Notice that, absent heterogeneity across inventor's knowledge capital, the allocation of inventors across firms would be irrelevant for research productivity and growth.

rate $\psi > 0$, in which case they catch up with the market leader.[9] Finally, small firms do not innovate, but keep a constant distance to the leader, denoted by $\alpha \equiv \frac{q_{cjt}}{\max\{q_{ijt}, q_{-ijt}\}} \in (0,1)$.

Inventors Inventors can be either *unattached* (i.e. not working for a firm) or *attached*.[10] Regardless of this status, the knowledge capital of the inventor depreciates (e.g. the inventor loses the patent on the κ-th innovation) at some exogenous rate $\delta > 0$, in which case her state goes from κ to $\max(\kappa - 1, \underline{\kappa})$. Moreover, the inventor finds an innovation, advancing κ to $\min(\kappa + 1, \overline{\kappa})$, at Poisson rate z and cost $\chi z^{\phi} \boldsymbol{Y}$, paid in units of final good, where $\chi > 0$. The inventor must finance this cost on her own. As explained above, if the inventor is attached to a firm, the firm chooses the rate with which to commercialize her idea.

Search and matching Large firms and inventors match in a frictional labor market.[11] The labor market is segmented by inventor type, κ. At each point in time, each large firm is matched with at most one inventor.[12] Only unattached inventors search for employment, while firms can keep searching for inventors while they are matched. Search is directed on both the inventor and the firm side: at every point in time, firms announce wage contracts in order to attract unattached inventors, and unattached inventors observe these contract and direct their search optimally to the wages offered by firms.

Contracts are complete, fully state-contingent and long-term, with full commitment on the firm's side and no commitment on the inventor's side. Given the assumption that inventors are risk-neutral, a sufficient statistic for each contract is the net present value that the firm promises to the inventor, which we denote by E. Therefore, each segment κ of the labor market can be described as a continuum of submarkets indexed by the value E that firms promise to inventors.

The measure of matches that take place in submarket (κ, E) per unit of time is given by $\mathtt{M}\big(\mathtt{F}_\kappa(E), \mathtt{U}_\kappa(E)\big)$, where $\mathtt{M} : \mathbb{R}^2_+ \to \mathbb{R}_+$ is a constant-returns-to-scale matching function, $\mathtt{F}_\kappa(E)$ is the measure of large firms posting contracts in submarket (κ, E), and $\mathtt{U}_\kappa(E)$ is the measure of unattached inventors searching in submarket (κ, E). Defining *market tightness* by $\theta_\kappa(E) \equiv \mathtt{F}_\kappa(E)/\mathtt{U}_\kappa(E)$, inventors match with a firm at Poisson rate $\mu(\theta_\kappa(E)) \equiv \mathtt{M}(\theta_\kappa(E), 1)$,

[9] In our model's Balanced Growth Path (BGP) equilibrium, this catch-up technology is necessary for the existence of an invariant distribution of industries.

[10] We prefer to use the term "unattached" over "unemployed" because when we bring the model to the data, "unattached" inventors in the model will not necessarily be mapped to inventors that do not have an employer in the data. See Section 4.2 for details.

[11] Small firms in the competitive fringe do not innovate, and therefore do not access the market for inventors.

[12] Further, we assume no firm can operate without an inventor. Thus, the measure of attached inventors is equal to the measure of large firms (equal to 2, as there is a unit-measure of industries and two large firms per industry).

while firms find an inventor of type κ with Poisson intensity $\eta(\theta_\kappa(E)) \equiv \mathtt{M}(1, 1/\theta_\kappa(E))$, so that $\mu(\theta) = \theta\eta(\theta)$.[13]

Entry Finally, we introduce endogenous entry and exit of large firms. Entry is directed at each labor market segment κ. Every period, a so-called "potential entrant" from a pool of (exogenous) measure $\mathtt{m}^P > 0$ pays a flow cost $c^e_\kappa \boldsymbol{Y}_t$ to enter submarket κ, where $c^e_\kappa > 0$, for each $\kappa \in \mathbb{K}$, are parameters. Paying this cost grants the right to offer a set of promised values to unattached inventors of type κ, which we denote by E^E. We assume these promised values are contingent on the state of the industry where the entrant will land, which makes entry effectively directed at the intermediate good industry level.

A successful entrant hires an inventor of type κ and replaces the follower in its industry.[14] The displaced firm exits the industry and its inventor becomes unattached. We assume there is free entry in every labor market segment κ.

3.2 Equilibrium

3.2.1 Households

The household's dynamic problem leads to the standard Euler equation for consumption:

$$\frac{\partial_t \boldsymbol{C}_t}{\boldsymbol{C}_t} = r_t - \rho. \tag{9}$$

3.2.2 Firms' Static Pricing Decisions

As is standard in the literature, we solve for a Markov Perfect Equilibrium (MPE) in which we solve first for the pricing decisions that result from the solution to a Bertrand game that is played statically to maximize flow profits. Given these, we then move to the dynamic choices, where large firms must strategically choose implementation rates for its incumbent inventor's idea, as well as hiring rates for the arrival of new inventors. Small firms, by contrast, price at marginal cost, have no profits and make no innovation decisions.

The cost-minimization problem of the representative final good producer generates the following demand schedule for good j produced by firm $f \in \{i, -i, c\}$:

[13] Further, we assume: (i) $\mu(\theta)$ is increasing and concave in θ; (ii) $\eta(\theta)$ is decreasing and convex in θ; (iii) standard Inada conditions apply, i.e. $\mu(0) = \lim_{\theta \to +\infty} \eta(\theta) = 0$, and $\lim_{\theta \to +\infty} \mu(\theta) = \lim_{\theta \to 0} \eta(\theta) = +\infty$.

[14] For neck-to-neck industries, the identity of the exiting firm is determined by a coin flip.

$$y_{fjt} = \left(\frac{p_{fjt}}{P_{jt}}\right)^{\frac{1}{\sigma-1}} \frac{\boldsymbol{Y}_t}{P_{jt}}, \quad \text{where } P_{jt} = \left(\sum_{f=i,-i,c} p_{fjt}^{\frac{\sigma}{\sigma-1}}\right)^{\frac{\sigma-1}{\sigma}}, \tag{10}$$

where p_{fjt} is the price of firm f, and P_{jt} is the ideal price index of industry j, holding $P_{jt}Y_{jt} = \sum_f p_{fjt}y_{fjt} = \boldsymbol{Y}_t$. Taking its demand schedule and the competitor's price as given, firm f sets a markup M_{fjt} over its marginal cost w_t^P/q_{fjt}, equal to:

$$M_{fjt} \equiv \frac{p_{fjt}}{w_t^P/q_{fjt}} = \frac{1-\sigma s_{fjt}}{\sigma(1-s_{fjt})}, \tag{11}$$

where $s_{fjt} = \frac{p_{fjt}y_{fjt}}{P_{jt}Y_{jt}} = \left(\frac{p_{fjt}}{P_{jt}}\right)^{\frac{\sigma}{\sigma-1}}$ denotes firm f's market share. A large firm's markup is a decreasing function of its demand's elasticity and, as in other models inspired by Atkeson and Burstein (2008), this elasticity decreases in equilibrium with the firm's sales share of the industry. Large firms' static profits then equal:

$$\Pi_{fjt} = s_{fjt}(1-M_{fjt}^{-1})\boldsymbol{Y}_t = \frac{s_{fjt}(1-\sigma)}{1-\sigma s_{fjt}}\boldsymbol{Y}_t. \tag{12}$$

As market share is only a function of the technology gap n_{ijt}, this serves as a sufficient statistic to evaluate firms' static payoffs.[15] Therefore, we may write $\Pi_{ijt} = \pi(n_{ijt})\boldsymbol{Y}_t$. Small firms, by contrast, price at marginal cost, so $M_{cjt} = 1$ and $\Pi_{cjt} = 0$. The fringe as a whole has a market share of $s_{cjt} = 1 - s_{ijt} - s_{-ijt}$.

3.2.3 Dynamic Equilibrium Conditions

Large firms' dynamic problem has three state variables. The first one is n, the technology gap with respect to its competitor within the industry, which determines the firm's static profits. The second state is κ, the type of inventor that the firm employs, which determines the gains from innovation. Finally, since innovation decisions are strategic, the firm needs to take the policies of its competitor into account. As these policies depend on the competitors' inventor type (denoted κ^-), this is also a state variable for the firm.

Recursive contracts A dynamic contract offered at time t by a firm with current technology gap n_t who is employing an inventor with knowledge capital κ_t, and facing a competitor

[15] To see this, note that we can write any firm f's market share as $s_{fjt} = \left[1 + \sum_{i\neq f}\left(\frac{p_{ijt}}{p_{fjt}}\right)^{\frac{\sigma}{\sigma-1}}\right]^{-1}$. Suppose firm i is the leader without loss of generality. Then, using the optimal pricing conditions we obtain $\frac{p_{-ijt}}{p_{ijt}} = \lambda^{n_{ijt}}\frac{1-\sigma s_{-ijt}}{1-\sigma s_{ijt}}\frac{1-s_{ijt}}{1-s_{-ijt}}$ and $\frac{p_{cjt}}{p_{ijt}} = \frac{1}{\alpha}\frac{\sigma(1-s_{ijt})}{1-\sigma s_{ijt}}$. Thus, market share, markup, and flow profits only depend on n_{ijt}.

employing an inventor of type κ_t^-, specifies a state-contingent wage trajectory $\widehat{w}_t\big(n_\tau, \kappa_\tau, \kappa_\tau^- : \tau \in [t, t+T]\big)$, which determines a transfer, in units of the final good, from the firm to the inventor at every inventor tenure $T > 0$.

As we focus on the MPE of this economy, however, we can express this dynamic contract in its recursive form. A *recursive contract* specifies the current wage and some future promised utility, contingent on next period state. In a MPE, a recursive contract offered by a firm in state (n, κ, κ^-) in the market for inventors of productivity κ is the object:

$$\mathcal{C} = \Big(\widehat{w}, \big\{E'(n', \kappa', \kappa^{-\prime})\big\}\Big), \tag{13}$$

where $(n', \kappa', \kappa^{-\prime})$ is the continuation state, conditional on match survival, and $\big\{E'(n', \kappa', \kappa^{-\prime})\big\}$ is the countably finite set of continuation promises, one for each possible realization of the future state. Given this recursive property of the equilibrium, henceforth we drop time t subscripts unless otherwise needed.

Unattached inventors Unattached inventors direct their job search towards the most attractive offers. Thus, the value of a type-κ inventor while unattached equals $\boldsymbol{U}_\kappa = \max_E \big(U_\kappa(E)\big)$, where $U_\kappa(E)$ is the value of searching in submarket E for a type-κ inventor. The latter is the solution to the Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned} rU_\kappa(E) - \partial_t U_\kappa(E) &= \mu(\theta_\kappa(E)) \max\Big(E - U_\kappa(E), 0\Big) \\ &\quad + \max_z \Big\{ z\Big(U_{\min(\kappa+1,\overline{\kappa})}(E) - U_\kappa(E)\Big) - \chi z^\phi \boldsymbol{Y} \Big\} + \delta\Big(U_{\max(\kappa-1,\underline{\kappa})}(E) - U_\kappa(E)\Big). \end{aligned} \tag{14}$$

The first-order condition for innovating while unattached is:

$$z_\kappa^U(E) = \left(\frac{U_{\min(\kappa+1,\overline{\kappa})}(E) - U_\kappa(E)}{\chi\phi\boldsymbol{Y}}\right)^{\frac{1}{\phi-1}}. \tag{15}$$

Free-entry of inventors of type κ implies the complementary-slackness condition:

$$\forall (E, \kappa) \in \mathbb{R}_+ \times \mathbb{K} : \quad U_\kappa(E) \le \boldsymbol{U}_\kappa, \text{ with equality if, and only if, } \mu\big(\theta_\kappa(E)\big) > 0. \tag{16}$$

In words, since inventors can choose where to search, the value of being unattached must be equal in all markets where job finding is positive. This gives rise to an indifference condition on the inventor's side: since unattached inventors choose the best market to search in, all active markets must be equally attractive ex ante from the unattached inventor's point of view. Replacing $U_\kappa(E)$ by $\boldsymbol{U}_\kappa$ for all active markets E and all levels of knowledge capital κ allows us to write the value of being unattached and type κ as:

$$r\boldsymbol{U}_\kappa - \partial_t \boldsymbol{U}_\kappa = \mu(\theta_\kappa(E))\,(E - \boldsymbol{U}_\kappa) + \boldsymbol{Z}_\kappa, \tag{17}$$

we have used the short-hand notation $\boldsymbol{Z}_\kappa \equiv \delta\big(\boldsymbol{U}_{\max(\kappa-1,\underline{\kappa})} - \boldsymbol{U}_\kappa\big) + \chi(\phi-1)(z_\kappa^U(E))^\phi \boldsymbol{Y}$. From here, we can obtain the *equilibrium market tightness mapping* used by inventors and firms in the market for inventors of type κ, denoted by $\theta_\kappa : (E, \boldsymbol{U}_\kappa) \to \mathbb{R}_+$, and given by:

$$\theta_\kappa(E) = \mu^{-1}\Big(\boldsymbol{\Phi}_\kappa(E)\Big), \qquad \text{where } \boldsymbol{\Phi}_\kappa(E) \equiv \frac{r\boldsymbol{U}_\kappa - \partial_t\boldsymbol{U}_\kappa - \boldsymbol{Z}_\kappa}{E - \boldsymbol{U}_\kappa}. \tag{18}$$

Market tightness is a decreasing function of E: more ex-post profitable offers attract a larger measure of inventors. Thus, in equilibrium, firms design contracts for which a low meeting rate for inventors is compensated with higher promised value.[16]

Attached inventors and firms Let us denote the value of an inventor of type κ that is attached to a firm in state (n, κ, κ^-) by $\boldsymbol{E}(n, \kappa, \kappa^-)$.[17] Denote this firm's value by $\boldsymbol{V}(n, \kappa, \kappa^-, E)$.[18] For brevity, we relegate the HJB equations describing these values to Appendix A.1 (see equations (A.2) and (A.4), respectively).

The dynamic problem of the firm has three choice variables: the current wage, $\widehat{w}$; the set of promised values $\{E'(n', \kappa', \kappa^{-\prime})\}$, one for each possible next-stage realization of the state; and the rate of implementation of the type-κ inventor's idea, x. Importantly, the firm's choices are strategic, taking as given the competitor's policies —both those of the other large incumbent in the industry, as well as those of the potential entrants in the industry.

In choosing the optimal contract, the firm is constrained by promise-keeping and inventor's participation constraints. Promise-keeping arises from the assumption of commitment on the firm side, which implies that contracts must deliver at least the promised utility to the worker, i.e., $\boldsymbol{E}(n, \kappa, \kappa^-) \geq E$. The participation constraint arises from the assumption of no commitment on the inventor side: the value that the inventor obtains in any future state cannot be lower than its outside option, or else the inventor would voluntarily quit the firm. Formally, $E'(n', \kappa', \kappa^{-\prime}) \geq \boldsymbol{U}_{\kappa'}$, for all possible continuation states $(n', \kappa', \kappa^{-\prime})$.

[16] Market tightness is also a function of the value of being unattached, $\boldsymbol{U}_\kappa$, which is determined in equilibrium: a better outside option for unattached inventors makes jobs relatively less attractive ex-ante. In fact, in a BGP, we will obtain that $\partial_t \boldsymbol{U}_\kappa > 0$, so promises will have to keep up with raising outside options to keep inventors interested in employment.

[17] Notice our notation convention: E denotes the firm's outstanding promise, while $\boldsymbol{E}$ (in bold) denotes the attached inventor's realized value.

[18] Notice that a firm's outstanding promise E is a state variable because of our commitment assumption, as we discuss below.

Joint surplus problem While the solution to the firm's and the inventor's problems, written in equations (A.2) and (A.4) in Appendix A.1, seems hard to characterize, our assumptions allow us to solve a simpler and equivalent problem instead: the maximization of the *joint surplus* between a firm and its inventor. Appendix A.2 shows this equivalence formally. In what follows, we therefore focus on the solution of the joint surplus problem.

Define the (gross) joint surplus as the sum of the firm's and the inventor's match values:

$$\boldsymbol{\Omega}(n,\kappa,\kappa^-,E) \equiv \boldsymbol{V}(n,\kappa,\kappa^-,E) + \boldsymbol{E}(n,\kappa,\kappa^-). \tag{19}$$

As we argue in Appendix A.2, two observations can be made of the joint surplus $\boldsymbol{\Omega}$. First, the promise-keeping constraint must hold with equality in equilibrium, i.e., $\boldsymbol{E}(n,\kappa,\kappa^-) = E$. This is because, by monotonicity of preferences, the firm will always choose to offer the lowest possible value to the inventor so that initial promises are still honored. Second, the joint surplus is independent of the wage $\widehat{w}$ and the promised value E is equilibrium, as these are pure utility transfers between two linear-utility agents that agree on how to evaluate payoffs. This allows us to write $\boldsymbol{\Omega}(n,\kappa,\kappa^-,E)$ as just $\boldsymbol{\Omega}(n,\kappa,\kappa^-)$.

3.2.4 BGP equilibrium

With these observations in place, to make progress we focus on a Balanced Growth Path (BGP), defined as a MPE in which all aggregate variables grow at a common and time-invariant rate, denoted $g > 0$.

To solve for the BGP, we guess-and-verify that there exist *time-invariant* functions $\boldsymbol{v}$, $\boldsymbol{e}$, $\boldsymbol{u}$, and w, such that, for all n, κ, κ^- and E, the following identities hold: (i) $\boldsymbol{V}(n,\kappa,\kappa^-,E) = \boldsymbol{v}(n,\kappa,\kappa^-,E/\boldsymbol{Y})\boldsymbol{Y}$, (ii) $\boldsymbol{E}(n,\kappa,\kappa^-) = \boldsymbol{e}(n,\kappa,\kappa^-)\boldsymbol{Y}$, (iii) $\boldsymbol{U}_\kappa = \boldsymbol{u}_\kappa\boldsymbol{Y}$, and (iv) $\widehat{w} = w\boldsymbol{Y}$. Together with our observations from the previous paragraph, this means that we can write the joint surplus normalized by GDP as:

$$\boldsymbol{\omega}(n,\kappa,\kappa^-) \equiv \frac{\boldsymbol{\Omega}(n,\kappa,\kappa^-)}{\boldsymbol{Y}} = \boldsymbol{v}(n,\kappa,\kappa^-,e) + e. \tag{20}$$

with $e \equiv E/\boldsymbol{Y}$.[19] Once again, the key observation is that the normalized joint surplus is independent of the outstanding (normalized) promise, $e \equiv E/\boldsymbol{Y}$, and depends only on the *new* promises made by the firm to the inventor, $e' \equiv E'/\boldsymbol{Y}$. Precisely, in Appendix A.2, we

[19] Importantly, along the BGP, hiring rates grow linearly with total output as well. Using the Euler equation $r = g + \rho$, equation (17) now reads $\rho \boldsymbol{u}_\kappa = b + \mu\big(\theta_\kappa(E)\big)\big(E - \boldsymbol{u}_\kappa\big) + \boldsymbol{z}_\kappa$, where $\boldsymbol{z}_\kappa \equiv \boldsymbol{Z}_\kappa/\boldsymbol{Y}$ is constant as $z_\kappa^U = \left(\frac{\boldsymbol{u}_{\min(\kappa+1,\bar{\kappa})} - \boldsymbol{u}_\kappa}{\chi\phi}\right)^{\frac{1}{\phi-1}}$ is constant. Therefore, we have $\theta_\kappa(e) = \mu^{-1}\left(\frac{\rho\boldsymbol{u}_\kappa - b - \boldsymbol{z}_\kappa}{e - \boldsymbol{u}_\kappa}\right)$. Finally, as the promise-keeping constraint must bind with equality, and the value of the inventor grows linearly in aggregate output, it must be that $e \equiv E/\boldsymbol{Y}$ is indeed constant.

show that the normalized joint surplus satisfies the following HJB equation (which, as seen in the following equation, indeed does not depend on e):[20]

$$
\begin{aligned}
\rho\boldsymbol{\omega}(n,\kappa,\kappa^-) = \max_{\substack{\{e'(n',\kappa',\kappa^{-\prime})\} \\ x>0, z>0}} \Bigg\{ & \underbrace{\pi(n)}_{\text{Flow profits}} - \underbrace{\chi z^{\phi}}_{\substack{\text{Innovation} \\ \text{costs}}} - \underbrace{\xi x^{\phi}}_{\substack{\text{Implementation} \\ \text{costs}}} \qquad (21) \\
\textit{Knowledge capital depreciates} \quad & + \delta\Big(\boldsymbol{\omega}\Big(n, \max(\kappa-1,\underline{\kappa}), \max(\kappa^- -1, \underline{\kappa})\Big) - \boldsymbol{\omega}(n,\kappa,\kappa^-)\Big) \\
\textit{Firm's inventor innovates} \quad & + z\Big(\boldsymbol{\omega}\Big(n, \min(\kappa+1,\overline{\kappa}), \kappa^-\Big) - \boldsymbol{\omega}(n,\kappa,\kappa^-)\Big) \\
\textit{Firm hires new inventor} \quad & + \sum_{\kappa'\in\mathbb{K}} \zeta_{\kappa'}\Big(e'(n,\kappa',\kappa^-)\Big)\Big(\boldsymbol{\omega}(n,\kappa',\kappa^-) - \boldsymbol{\omega}(n,\kappa,\kappa^-) + \boldsymbol{u}_\kappa - e'(n,\kappa',\kappa^-)\Big) \\
\textit{Competitor's inventor innovates} \quad & + \widetilde{z}\Big(\boldsymbol{\omega}\Big(n,\kappa,\min(\kappa^- +1,\overline{\kappa})\Big) - \boldsymbol{\omega}(n,\kappa,\kappa^-)\Big) \\
\textit{Competitor hires new inventor} \quad & + \sum_{\kappa^{-\prime}\in\mathbb{K}} \zeta_{\kappa^{-\prime}}\Big(\widetilde{e}'(-n,\kappa^{-\prime},\kappa)\Big)\Big(\boldsymbol{\omega}(n,\kappa,\kappa^{-\prime}) - \boldsymbol{\omega}(n,\kappa,\kappa^-)\Big) \\
\textit{Firm implements innovation} \quad & + x\Big(\boldsymbol{\omega}\Big(\min(n+\kappa,\overline{n}),\kappa,\kappa^-\Big) - \boldsymbol{\omega}(n,\kappa,\kappa^-)\Big) \\
\textit{Competitor implements innovation} \quad & + \widetilde{x}\Big(\boldsymbol{\omega}\Big(\max(n-\kappa^-,-\overline{n}),\kappa,\kappa^-\Big) - \boldsymbol{\omega}(n,\kappa,\kappa^-)\Big) \\
\textit{Follower catches up to leader} \quad & + \psi\Big(\boldsymbol{\omega}(0,\kappa,\kappa^-) - \boldsymbol{\omega}(n,\kappa,\kappa^-)\Big) \\
\textit{Potential entrant enters} \quad & + \mathtt{m}^P \sum_{\kappa'\in\mathbb{K}} \boldsymbol{\omega}^E(n,\kappa,\kappa^-;\kappa') \Bigg\}
\end{aligned}
$$

subject to

$$
e'(n',\kappa',\kappa^{-\prime}) \geq \boldsymbol{u}_{\kappa'} \quad \forall(n',\kappa',\kappa^{-\prime}), \tag{22}
$$

where we have used the short-hand notation $\zeta_\kappa(e) \equiv \eta \circ \mu^{-1}(\boldsymbol{\Phi}_\kappa(e))$ for the job-filling rate. Moreover, we have defined the change in the value from an entry event (last term in

[20]In this equation, competitor policies (both those of the other incumbent firm as well as those of the potential entrant) are indicated with tildes, e.g. $\widetilde{z}$ is the innovation policy of the firm's competitor. These policies are taken as given by the firm when making decisions.

equation (21)) as:

$$\omega^E(n,\kappa,\kappa^-;\kappa') \equiv \begin{cases} \zeta_{\kappa'}\Big(\widetilde{e}^E_{\kappa'}(n,\kappa,\kappa^-)\Big)\Big(\omega(n,\kappa,\kappa') - \omega(n,\kappa,\kappa^-)\Big) & \text{if } n > 0 \\ \zeta_{\kappa'}\Big(\widetilde{e}^E_{\kappa'}(-n,\kappa,\kappa^-)\Big)\Big(u_\kappa - \omega(n,\kappa,\kappa^-)\Big) & \text{if } n < 0 \\ \zeta_{\kappa'}\Big(\widetilde{e}^E_{\kappa'}(0,\kappa,\kappa^-)\Big)\Big(\frac{1}{2}\omega(0,\kappa,\kappa') + \frac{1}{2}u_\kappa - \omega(0,\kappa,\kappa^-)\Big) & \text{if } n = 0. \end{cases}$$

Equation (21) tells us that the joint surplus is composed of the flow surplus (first line), plus the changes in joint surplus value due to inventor separation, inventor innovation, own idea implementation and human capital depreciation, plus the changes due to the competitor hiring, innovating, and implementing the idea of its inventor. The last three lines consider entry of a potential entrant, with implications that depend on the firm's position in the industry.

Optimal policies As stated before, the firm's problem (written in Appendix A.1) and the joint surplus problem (equation (21)) are equivalent. Appendix A.2 proves this result formally. This implies that the contract that maximizes the firm's value can be found by maximizing the joint surplus. Specifically, as the joint surplus is invariant to the wage and the outstanding promise of the firm, we can solve for the optimal contract in two stages: (i) we solve problem (21) to find the optimal future promises and innovation policies; (ii) we find wages residually by ensuring that the promise-keeping constraint is binding and the worker-participation constraint is satisfied at all points of the state space.

Taking first-order conditions in (21), the optimal implementation rate of a firm in state n that employs an inventor of type κ and faces a competitor whose inventor is of type κ^- is:

$$x_{n,\kappa,\kappa^-} = \left(\frac{\omega\big(\min(n+\kappa,\overline{n}),\kappa,\kappa^-\big) - \omega\big(n,\kappa,\kappa^-\big)}{\xi\phi}\right)^{\frac{1}{\phi-1}}. \tag{23}$$

The optimal innovation rate of the inventor that is attached to this firm is:

$$z_{n,\kappa,\kappa^-} = \left(\frac{\omega\big(n,\min(\kappa+1,\overline{\kappa}),\kappa^-\big) - \omega\big(n,\kappa,\kappa^-\big)}{\chi\phi}\right)^{\frac{1}{\phi-1}}. \tag{24}$$

Moreover, taking the first order condition with respect to $e'(n,\kappa',\kappa^-)$, we obtain:

$$\frac{\partial\zeta_{\kappa'}(e)}{\partial e}\bigg|_{e=e'(n,\kappa',\kappa^-)}\Big(\omega\big(n,\kappa',\kappa^-\big) - \omega\big(n,\kappa,\kappa^-\big) + u_\kappa\Big) = \ \ldots \tag{25}$$

$$\frac{\partial \zeta_{\kappa'}(e)}{\partial e}\bigg|_{e=e'(n,\kappa',\kappa^-)} e'\left(n,\kappa',\kappa^-\right) + \zeta_{\kappa'}\Big(e'\left(n,\kappa',\kappa^-\right)\Big),$$

which equates the expected marginal gain in joint surplus value from hiring a new inventor of type κ' and letting the incumbent type-κ inventor separate (left-hand side), to the marginal cost from paying the promised value to the new hire (right-hand side).

From (21), we can solve for the joint surplus $\boldsymbol{\omega}(n,\kappa,\kappa^-)$ at every state $(n,\kappa,\kappa^-) \in \{-\overline{n},\ldots,0,\ldots,\overline{n}\} \times \mathbb{K} \times \mathbb{K}$ using value function iteration. The resulting allocation will be a function, however, of the equilibrium inventor outside options, $\{\boldsymbol{u}_\kappa\}_{\kappa\in\mathbb{K}}$. To pin these down in equilibrium, we turn to the potential entrants' problem.

Potential entrants At any given point in time, the state of an industry can be summarized by (m,κ^L,κ^F), where $m \in \{0,1,\ldots,\overline{n}\}$ denotes the gap between leader and follower; $\kappa^L \in \mathbb{K}$ denotes the inventor type attached to the leader; and $\kappa^F \in \mathbb{K}$ denotes the inventor type attached to the follower.[21]

Every period, potential entrants pay a flow cost $c^e_\kappa \boldsymbol{Y}$ to enter labor market segment κ and offer a set of promised values, $\{e^E_\kappa(m,\kappa^L,\kappa^F)\}$, to unattached inventors. At rate $\zeta_\kappa\big(e^E_\kappa(m,\kappa^L,\kappa^F)\big)$, a potential entrant hires an inventor of type κ and replaces the follower in a random industry from state (m,κ^L,κ^F).[22] Therefore the value, per unit of output, of a potential entrant aiming at labor market segment κ, denoted $\boldsymbol{v}^E_\kappa$, satisfies:

$$\begin{aligned}
\rho \boldsymbol{v}^E_\kappa \; &= -c^e_\kappa + \max_{\{e^E_\kappa(m,\kappa^L,\kappa^F)\}} \Bigg\{ \sum_{\kappa^L\in\mathbb{K}} \sum_{\kappa^F\in\mathbb{K}} \zeta_\kappa\Big(e^E_\kappa(0,\kappa^L,\kappa^F)\Big) \frac{1}{2} \sum_{h=F,L} \boldsymbol{v}\Big(0,\kappa,\kappa^h, e^E_\kappa(0,\kappa^L,\kappa^F)\Big) \\
&\quad + \sum_{m=1}^{\overline{n}} \sum_{\kappa^L\in\mathbb{K}} \sum_{\kappa^F\in\mathbb{K}} \zeta_\kappa\Big(e^E_\kappa(m,\kappa^L,\kappa^F)\Big) \boldsymbol{v}\Big(-m,\kappa,\kappa^L, e^E_\kappa(m,\kappa^L,\kappa^F)\Big) - \boldsymbol{v}^E_\kappa \Bigg\} \\
&\qquad \text{subject to } \; e^E_\kappa(m,\kappa^L,\kappa^F) \geq \boldsymbol{u}_\kappa, \quad \forall (m,\kappa^L,\kappa^F).
\end{aligned} \tag{26}$$

In an equilibrium with positive entry in all market segments, we must have $\boldsymbol{v}^E_\kappa = 0$, for all $\kappa \in \mathbb{K}$. Then, using equation (20), we can write the free-entry condition in market $\kappa \in \{\underline{\kappa},\ldots,\overline{\kappa}\}$ in terms of joint surplus as:

$$c^e_\kappa = \max_{\{e^E_\kappa(m,\kappa^L,\kappa^F)\}} \Bigg\{ \sum_{\kappa^L\in\mathbb{K}} \sum_{\kappa^F\in\mathbb{K}} \zeta_\kappa\Big(e^E_\kappa(0,\kappa^L,\kappa^F)\Big) \left[\frac{1}{2} \sum_{h=F,L} \boldsymbol{\omega}(0,\kappa,\kappa^h) - e^E_\kappa(0,\kappa^L,\kappa^F)\right]$$

[21] Notice that here we make a distinction between n, the technology gap from a given firm's point of view (and therefore an individual firm's state variable), and m, the number of steps that the leader is ahead of the follower in the industry (and therefore an industry-level state variable).

[22] If $m = 0$, then the exiting firm is determined by a coin flip.

$$+\sum_{m=1}^{\overline{n}}\sum_{\kappa^L\in\mathbb{K}}\sum_{\kappa^F\in\mathbb{K}}\zeta_\kappa\left(e^E_\kappa(m,\kappa^L,\kappa^F)\right)\left(\boldsymbol{\omega}(-m,\kappa,\kappa^L)-e^E_\kappa(m,\kappa^L,\kappa^F)\right)\Bigg\}.$$

These free entry conditions allow us to pin down the values of unattached inventors in equilibrium, $\{\boldsymbol{u}_\kappa\}_{\kappa\in\mathbb{K}}$.

3.2.5 Closing the model

To close the equilibrium characterization, we impose market clearing conditions. The market-clearing condition for production labor, $1=\int_0^1(l_{ijt}+l_{-ijt}+l_{cjt})dj$, gives rise to:

$$\boldsymbol{M}_t=\left[\int_0^1\left(\sum_{f=i,-i,c}s_{fjt}M_{fjt}^{-1}\right)dj\right]^{-1}, \tag{27}$$

where $\boldsymbol{M}_t\equiv\frac{\boldsymbol{Y}_t}{w_t^P}$ is the aggregate markup, or the inverse of the aggregate (production) labor share.[23] Final output is used for household consumption and to pay for innovation and implementation costs, inventor's consumption while unattached, and entry costs. Therefore, the resource constraint reads:

$$1=\frac{\boldsymbol{C}_t}{\boldsymbol{Y}_t}+\int_0^1\xi(x^\phi_{ijt}+x^\phi_{-ijt})\,dj+\int_0^1\chi(z^\phi_{ijt}+z^\phi_{-ijt})\,dj+\sum_{\kappa\in\mathbb{K}}\left[\varphi^U_{\kappa t}\left(\chi(z^U_{\kappa t})^\phi\right)+c^e_\kappa\mathtt{m}^P\right], \tag{28}$$

where $\varphi^U_{\kappa t}$ denotes the equilibrium measure of unattached inventors of type κ at time t.

To close the characterization of the equilibrium, we must find the equilibrium measures of industries in BGP. These can be found from two sets of flow equations that determine the evolution of the measures of industries, $\{\varphi_{m,\kappa^L,\kappa^F}\}$, and of unattached inventors, $\{\varphi^U_\kappa\}$. To save space, we report these equations in Appendix A.3.

3.2.6 Growth

We are now ready to derive the growth rate of the economy. In a BGP, output grows at the rate of the productivity frontier, g. In turn, the productivity frontier, denoted by $\boldsymbol{Q}_t$, is determined by the productivity of industry leaders, so $\boldsymbol{Q}_t=\exp\left(\int_0^1\ln(q^L_{jt})dj\right)$, where $q^L_{jt}\equiv\max\{q_{ijt},q_{-ijt},q_{cjt}\}$. Then, as we derive in Appendix A.4:

[23] As is typical of oligopolistic models with inelastic labor supply, the aggregate markup is a sales-weighted harmonic mean of firm-level markups. Notice that, in this economy, the aggregate markup is also equal to the cost-weighted average markup.

$$g = \sum_{m=0}^{\overline{n}} \sum_{\kappa^L \in \mathbb{K}} \sum_{\kappa^F \in \mathbb{K}} \varphi_{m,\kappa^L,\kappa^F} \left[\ln(\lambda^{\kappa^L}) x_{m,\kappa^L,\kappa^F} + \mathbb{1}_{\{\kappa^F > m\}} \ln(\lambda^{\kappa^F - m}) x_{-m,\kappa^F,\kappa^L} \right]. \qquad (29)$$

where x are the optimal implementation rates derived in equation (23).

Output growth depends on the arrival rate of innovations, which are due to both leaders' and followers' innovation efforts, weighted by the quality of their innovations. Different from the standard models, in our model innovation outcomes are heterogeneous. Notice, for example, that firms that lag behind their industry's leader contribute to growth if they employ a sufficiently productive inventor, one whose innovation would make the firm leap-frog the industry leader. Generally, therefore, the allocation of inventors, who are heterogeneous in the quality of their ideas, to firms, who are heterogeneous in their incentives to implement innovations, will have a first-order impact on growth.

3.3 Key Properties of the Model

Before turning to the calibration and our quantitative exercises, it is worth discussing some of the key properties of the equilibrium.[24]

3.3.1 Joint surplus, innovation and implementation policies

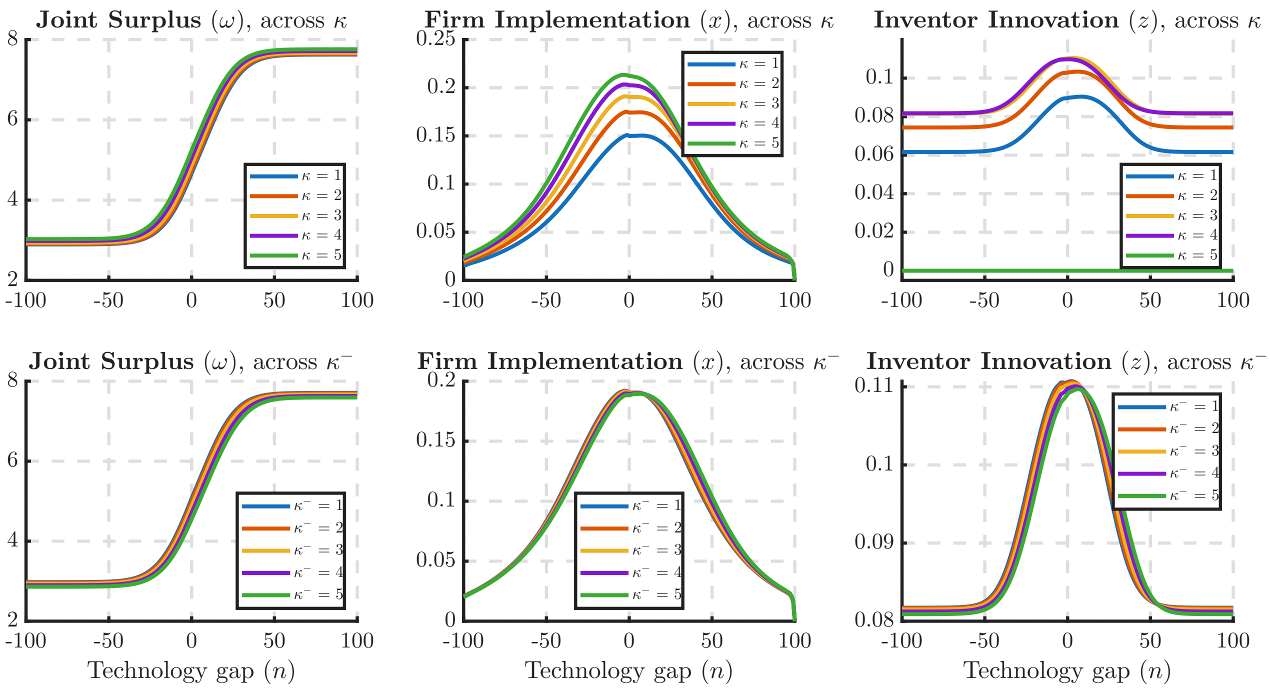
Figure 4 shows different cuts of the joint surplus $\omega(n, \kappa, \kappa^-)$, the innovation rate of attached inventors z_{n,κ,κ^-}, and the implementation rate of firms x_{n,κ,κ^-}, in equilibrium. For all plots, the horizontal axis is n, the technology gap from the perspective of a firm. The first row of plots show these objects for different values of the firm's inventor quality κ, for a fixed quality of the competitor firm's inventor, κ^-. The second row of plots fix the quality of the firm's inventor, and look at different levels of κ^- instead.

Certain properties of these value and policy functions become readily apparent. First, the joint surplus is increasing in the technology gap, featuring an S-shape relationship which is familiar from these type of models: the gains from innovation are greatest when the two competitor firms are closer to being neck-to-neck (a so-called "escape competition" effect). As firms' incentives to implement are monotonic in the change in joint surplus as per equation (23), the S-shape implies that firm implementation rates feature an inverted-U shape in the technology gap.

[24]The figures in this section have been drawn using our baseline calibration, which we discuss in Section 4. As explained in that section, we consider a Cobb-Douglas matching function, set $\overline{n} = 100$, and consider 5 different inventor productivity levels (normalizing $\underline{\kappa} = 1$, so that $\overline{\kappa} = 5$).

Second, and new to our model, there are important differences in levels depending on the quality of the inventors of both firms. For a given n, the joint surplus from the firm-inventor match is higher, other things equal, when the firm's *own* inventor is more talented, and when the *competitor*'s inventor is less talented. As a consequence, firm implementation rates increase (decrease) in the quality of the own (rival) inventor: firms that employ better inventors will, on average, implement their inventor's ideas more often.

Figure 4: Joint surplus, innovation rates by inventors and implementation rates by firms



Notes: This figure shows, as a function of the technology gap from the perspective of a firm (where $n > 0$ means leader and $n < 0$ means follower), the joint surplus (first column), the firm's optimal implementation rate of its inventor's idea (second column), and inventor's optimal innovation rate (third column). The first row plots these objects for different values of κ (the quality of the firm's own inventor), for a fixed κ^- (the quality of the rival firm's inventor). The second row plots the same objects for different values of κ^-, but now keeping κ fixed.

Third, the peak of the inverted-U in the firm implementation rate moves to the left as κ increases: firms employing better inventors can achieve higher productivity advancements by implementing their ideas at the same rate. However, the effect of the competitor employing a better inventor is negative for market followers that are close to their competitor and positive for market leaders with a small lead over their rival. Indeed, market leaders innovate more than they would if their competitor employed a worse inventor, because they understand that their small lead is at higher risk and they seek to defend it. By contrast, followers innovate less than they would if their competitor employed a worse inventor: when the leader employs a better inventor, it is harder for the follower to obtain market leadership. As a result, the follower gets discouraged and innovates less.

The third column of Figure 4 shows how the attached inventor internalizes these different forces in her innovation decisions. Better inventors put more effort into creating new ideas, as these are highly rewarded by the firms that use them.[25] However, inventors also react to the quality of the inventor that is attached to the competitor firm in the industry: just as was the case for firms and their implementation probabilities, an inventor will increase its innovation effort when faced with a better rival inventor if the firm that employs her is ahead of its rival, as in that case a successful implementation will improve the inventor's pay. However, inventors in laggard firms are discouraged to creating new inventions when faced with better rival inventors. In this sense, the inventor's and its firm's incentives are aligned, creating a sorting complementarity that is beneficial for growth.[26]

3.3.2 Job-filling rates

Figure 5 plots the average job-filling rate in an industry with distance $m = 0, 1, \ldots, \overline{n}$ between leader and follower in which firms are looking to hire an inventor of type κ'.[27]

For all industry states m, job-filling rates are higher for higher quality inventors, that is, on average firms put more effort into finding better inventors, and they do so by offering better promised values (i.e., wages).[28] Moreover, job-filling rates are lowest for firms that are far behind of their competitor and highest for firms that are close to neck-to-neck with their competitors, because of the escape-competition effect. However, job-filling rates are also relatively high for dominant firms in concentrated sectors: these firms promise high values to high-κ inventors, knowing that employing them will discourage innovation by their competitors.

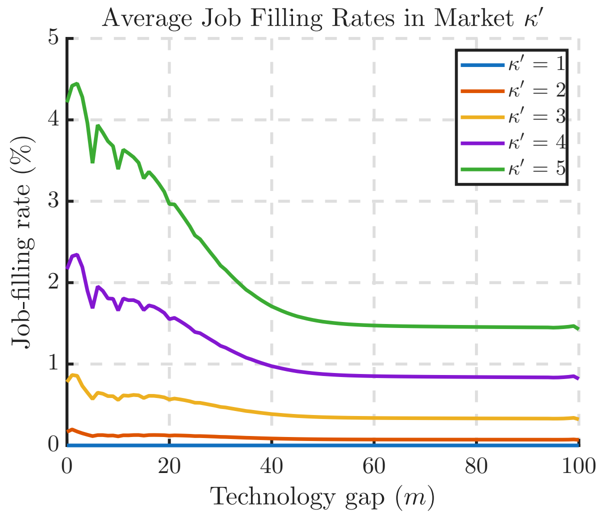
[25]The exception is, of course, inventors of the highest type (in this case, $\kappa = 5$), who are perfect free riders: once hired, because the firm has committed to its promises and there is no better inventor that can replace them, they choose not to exert any innovative effort at all.

[26]Indeed, once matched, firm and inventor both gain from higher match surplus.

[27]To be precise, to construct this figure, we first transform the Poisson rates $\zeta_{\kappa'}\left(e'(n, \kappa, \kappa^-)\right)$ into probabilities with the formula $1 - \exp(-\zeta)$. The resulting value gives us the probability that a firm, whose current inventor is of type κ and who keeps a gap $n = -\overline{n}, \ldots, 0, \ldots, \overline{n}$ with its competitor (whose inventor is of type κ^-), will see at least one hire of an inventor of type κ' within the year. Then, we average these firm-specific probabilities at the industry level using the invariant distribution of inventors across industries in the BGP equilibrium.

[28]Naturally, job-filling rates are zero for the lowest-quality inventor. As firms must always be matched to one inventor, they can choose to employ the worst inventor for free. Thus, they will put effort only in hiring inventors that dominate the worst possible type.

Figure 5: Average job-filling rate for an inventor of type κ', by industry state.



Notes: This figure plots, for each industry state $m = 0, 1, 2, \ldots$, the average probability of hiring an inventor of type κ' within a year.

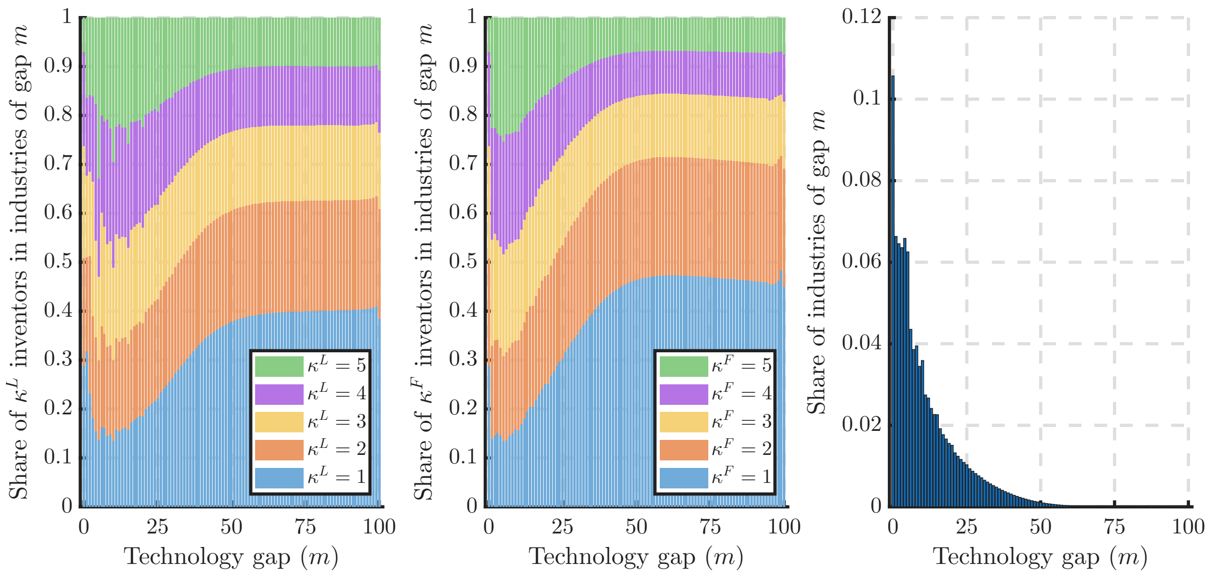
3.3.3 Distribution of inventors across industries

The differences in job-filling rates by industry state and across inventor types just discussed give rise to differences in the way that (attached) inventors of different productivities are ultimately allocated across industries.

To see this, Figure 6 offers different cuts of the invariant distribution of industries, $\varphi_{m,\kappa^L,\kappa^F}$, in the BGP equilibrium of the calibrated economy. The first panel shows the distribution of inventor types that are attached to leader firms by industry state $m = 0, 1, \ldots, \overline{n}$, as a share of all inventors in that state that are attached to a leader. The second panel shows the same, but for inventors attached to followers. In both cases, the same pattern emerges: the best inventors tend to sort into more competitive industries, where the incentives to implement ideas are strongest and, therefore, are well rewarded by firms through high wages. Qualitatively, this is the pattern that we saw in the data (recall Figure 3): top inventors, i.e. inventors at the top of the Q_{jt} distribution (recall equation (2)), are disproportionately allocated in low HHI (here, low m) industries. Indeed, we will use this as an untargeted moment for validation of the calibration strategy, which we describe in the next section.

The third panel of Figure 6 is the marginal distribution of industries across technology gaps (i.e. $\sum_{\kappa^L} \sum_{\kappa^F} \varphi_{m,\kappa^L,\kappa^F}$), showing that competitive (low m) industries are more abundant in equilibrium. Due to this fact, the concentration of knowledge in these industries, particularly the fact that highly talented inventors are more attracted to sectors in which firms are in stronger competition, is likely to have aggregate quantitative implications for

Figure 6: Equilibrium distribution of inventors across industry states.



Notes: This figure shows different cuts of the distribution of industries over industry states m, κ^L and κ^F in the BGP equilibrium. The left-most panel shows, for each industry state $m = 0, 1, 2, \ldots$, the share of inventors of each type relative to all inventors attached to leader firms in that industry state. The middle panel plots the same, except for inventor attached to follower firms. The third panel shows the overall distribution of industries across technology gaps, i.e. $\sum_{\kappa^L} \sum_{\kappa^F} \varphi_{m,\kappa^L,\kappa^F}$ for each m.

growth and welfare. Before analyzing these implications, we explain how we discipline the parameters of our model.

4 Calibration

4.1 Parameterization

We consider that any large firm can take at most $\overline{n} = 100$ productivity steps.[29] We normalize $\underline{\kappa} = 1$ and assume five inventor types, i.e. $\overline{\kappa} = 5$. Moreover, we assume a Cobb-Douglas matching function, $\mathtt{M}(\mathtt{F}, \mathtt{U}) = A\mathtt{F}^{\gamma}\mathtt{U}^{1-\gamma}$, where $\gamma > 0$ is the matching elasticity, and $A > 0$ is a matching efficiency parameter. As we show in Appendix A.5, with this functional form the value promised by a firm in state (n, κ, κ^-) to an inventor of type κ' is a weighted average of the prospective new inventor's outside option and the net gain in joint surplus that would result from this match, with weights given by the matching elasticity:

[29]The right panel of Figure 6 shows that $\overline{n} = 100$ is not a binding choice, in the sense that the share of industries in state $m = 100$ is almost zero.

$$e'_{\kappa}(n,\kappa',\kappa^-) = \gamma \boldsymbol{u}_{\kappa'} + (1-\gamma)\Big(\boldsymbol{\omega}\big(n,\kappa',\kappa^-\big) - \boldsymbol{\omega}\big(n,\kappa,\kappa^-\big) + \boldsymbol{u}_{\kappa}\Big). \tag{30}$$

Equations (A.21) to (A.24) in Appendix A.5 give, respectively, the equilibrium job-filling rate, the continuation values promised by potential entrants, the entry rates by labor market segment, and the free entry condition for each labor market $\kappa \in \{\underline{\kappa}, \ldots, \overline{\kappa}\}$.

4.2 Calibration Strategy

Model meets data To map firms between model and data, we consider an industry in the model to be the equivalent to a NAICS-4 digit sector in our Compustat sample. The leader and the follower in each product line correspond to the two largest firms in their Compustat industry in terms of sales, while the fringe is every other Compustat firm in that industry (if any). Attached inventors in the model correspond to those inventors in the USPTO data that are employed in either one of these two top Compustat firms. Unattached inventors, in contrast, are the remaining set of inventors in our USPTO-Compustat sample.

External identification Given this mapping between model and data, we calibrate the parameters in the BGP equilibrium of the model as follows. We set just three parameters externally: (ρ, σ, γ). We fix the discount rate to $\rho = 0.02$, which approximately corresponds to a discount factor of 97 percent annually and, together with a 2% target for the growth rate (see below), implies a 4% annualized interest rate. We set the CES parameter to $\sigma = 5/6$, implying an elasticity of substitution between intermediate-good varieties of $\epsilon = 6$, a mid-point number in the range of values that are customarily considered in the literature. Finally, for the elasticity of the matching function, we choose $\gamma = 0.5$, a standard value in the literature (e.g. Petrongolo and Pissarides (2001)).

Internally calibrated parameters There are 9 parameters left: $(\lambda, A, \chi, \xi, \delta, \psi, \alpha, \phi, \mathtt{m}^P)$.[30] These parameters are calibrated internally to minimize the distance between 9 model-generated moments and their empirical counterparts.[31] All of the data targets are computed as averages over the period 1980-2016.

[30]We avoid explicitly having to calibrate the entry costs $\{c^e_{\kappa}\}_{\kappa=\underline{\kappa}}^{\overline{\kappa}}$ by solving the model under a guess for the values of the unattached state, $\{\boldsymbol{u}_{\kappa}\}_{\kappa=\underline{\kappa}}^{\overline{\kappa}}$, and setting the entry costs "ex post" to the values that rationalize the free-entry condition given our guess. This saves us having to iterate over an (additional) entry loop in the numerical implementation of the model. Naturally, however, this loop cannot be avoided for the counterfactual experiments.

[31]We use a differential evolution algorithm to find the values for $\boldsymbol{\theta} \equiv (\lambda, A, \chi, \xi, \delta, \psi, \alpha, \phi, \mathtt{m}^P)$ that minimize the distance function $\sum_{m=1}^{9} \left(\frac{\text{Moment}_m(\text{Model},\boldsymbol{\theta}) - \text{Moment}_m(\text{Data})}{\text{Moment}_m(\text{Data})}\right)^2$.

First, we target the growth rate of GDP per capita, equal to 2% in the data (Jones, 2016). This helps us discipline the innovation step size λ. We obtain $\lambda = 1.0261$. As $\overline{\kappa} = 5$, this means that an implemented idea created by the most talented inventor in the economy advances a firm's productivity by 13.75%.

Second, to identify the matching efficiency parameter A, we target inventors' transition frequency from top-2 Compustat firms to non-top-2 Compustat firms, equal to 8.58% annually. Given the mapping to the data described above, in the model this corresponds to the attached-to-unattached (A-to-U) inventor flow rate, and will be used as our baseline measure of inventor mobility in our quantitative exercises to follow.

Third, the research and implementation cost shifter parameters, χ and ξ, are pinned down by (i) the share of USPTO inventors that file at least one patent in a given year, and (ii) the R&D spending share, computed as the average ratio of the R&D expenditures of the top 2 Compustat firms in each industry to their corresponding value added. We obtain 45.56% and 2.15%, respectively. In the model, the share of patenting inventors is computed as the average probability that an inventor, whether attached or unattached, produces an innovation in any given year. On the other hand, to compute the R&D spending share in the model, we include the costs from firm's idea implementation (from x), as well as those from inventor's idea creation (from z and z^U).

Fourth, to pin down the depreciation rate δ of inventor human capital, we target the average inventor productivity, computed in the data as described in Section 2.2. The model counterpart of this moment is the average quality κ, which we compute with the measure of attached inventors in the BGP equilibrium.

Fifth, to discipline the exogenous catch-up rate, ψ, and the fringe's distance to the leader, α, we target the cost-weighted average markup of the economy and the static misallocation losses from markup dispersion. We take as reference Edmond, Midrigan and Xu (2023), who report that for a cost-weighted average markup of 30%, output losses from static misallocation would amount to 1.8%.[32]

Sixth, to pin down the mass of potential entrants $\mathtt{m}^P$, we target the average firm entry rate from the Business Dynamics Statistics (BDS) of the U.S. Census Bureau. In the data, this number is 10.2% at the annual frequency (although it has trended downward over time). In the model, the entry rate is the average ratio of the measure of entrants per unit of time and the total measure of incumbent firms (equal to 2).

Finally, to calibrate the curvature parameter ϕ in the R&D cost function, we target an elasticity of innovation to R&D equal 0.5, a value that is commonly found in the empirical

[32]The implied (cost-weighted) standard deviation of markups in the calibration of the model is 0.161, in line with empirical estimates in the literature (De Loecker, Eeckhout and Unger (2020)).

literature (e.g. Acemoglu, Akcigit, Alp, Bloom and Kerr, 2018). In our model, this elasticity is not directly pinned down by the curvature parameter ϕ of the innovation cost function, as in models, because the innovation process involves two stages (idea creation by the inventor and implementation by the firm), each with different costs. Appendix A.6 shows how we compute this elasticity.

Table 1 reports the results of the calibration in terms of model fit, and provides the corresponding parameter values. The model matches all 9 targeted moments perfectly.

Table 1: Parameters and model fit

Parameter		**Value**	**Target/Source**	**Data**	**Model**
ρ	Discount rate	0.02	4% annual interest rate		
σ	CES parameter	5/6	Elasticity of substitution $= 6$		
γ	Matching elasticity	0.5	Petrongolo and Pissarides (2001)		
λ	Innovation step size	1.0261	Growth rate	0.02	0.02
A	Matching efficiency	0.1118	A-to-U inventors' transition rate	0.0858	0.0858
χ	Research cost shifter	913.129	Share of patenting inventors	0.4556	0.4556
ξ	Implementation cost shifter	116.562	R&D share (Compustat)	0.0215	0.0215
δ	Depreciation rate of κ	0.2004	Average inventor productivity	2.4062	2.4062
ψ	Exogenous catch-up rate	0.0309	Misallocation loss from markups	0.018	0.018
α	Fringe's distance to leader	0.5563	Cost-weighted markup	1.30	1.30
$\mathtt{m}^P$	Mass of potential entrants	0.0043	Firm entry rate	0.102	0.102
ϕ	R&D cost curvature	5.6459	Elasticity of innovation to R&D	0.5	0.5

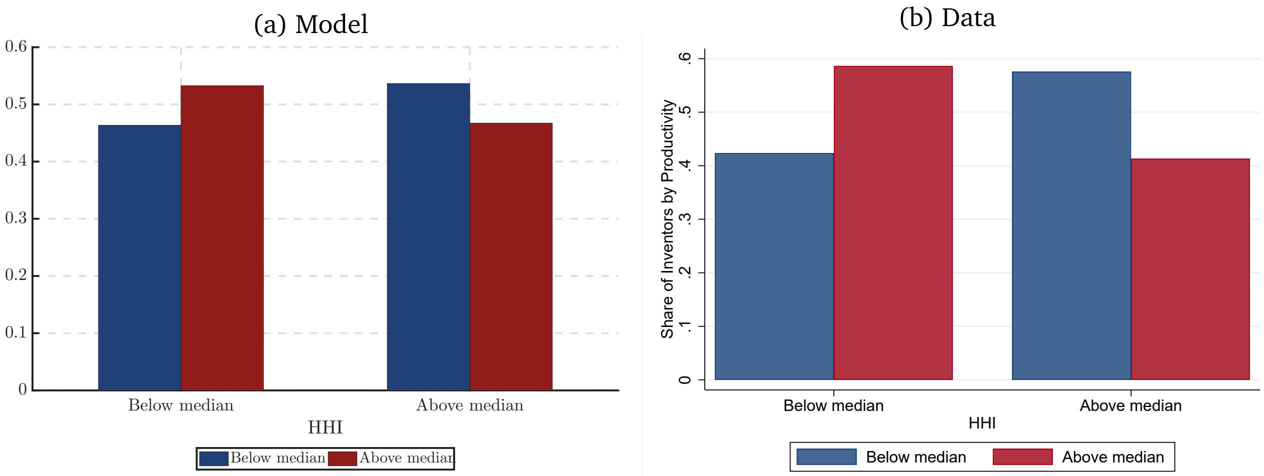
Note: The model period is one year. This table reports the values of the parameters estimated externally (first block) and internally (second block). The average markup is cost-weighted both in the data and in the model.

4.3 Validation: Inventor Sorting Across Industries

As a validation of our calibration strategy, we explore the inventor sorting patterns that the model delivers compared to those in the data. Figure 7 shows that our model delivers a distribution of inventors across industries that replicates the pattern observed in the data. In the model, as in the data, high (above-median) productivity inventors (the blue bars in the plots) are more likely to be attached to firms in competitive (below-median HHI) sectors than low (below-median) productivity inventors (the red bars), while low-productivity inventors are more likely to be attached to firms in more concentrated (above-median HHI) sectors.

Thus, even though not a calibration target, our model is able to roughly match the sorting patterns between inventors and industries that we see in the data. In the next

Figure 7: Distribution of inventors across industries



Notes: In panel (a), bars represent shares of below-median and above-median productivity (κ) inventors attached to firms in industries with below-median and above-median HHI index. In the data, we compute the distribution of inventor teams' productivity employed by the two largest firms in each NAICS-4 digit sector. Team productivity is computed as the average productivity of the inventors employed by the firm. In panel (b), bars represent shares of below-median and above-median productivity inventors attached to firms in industries with below-median and above-median HHI index.

section, we examine the aggregate implications of this sorting for growth and welfare.

5 Aggregate Effects of Inventor Sorting

We can now tackle the central research question of our paper: what are the macroeconomic effects of the misallocation of inventors within and across industries? And how can industrial policies correct for these inefficiencies?

5.1 The Role of Inventor Mobility

To gain some intuition, it is useful to start by considering the effects of inventor mobility (defined as the average attached-to-unattached inventor transition rate) on innovation and economic growth. This rate is an endogenous object, so we need an exogenous source of variation that can change it monotonically. One simple way to do this is to change the matching efficiency parameter, A, which proxies the degree of matching frictions in the labor market for inventors.[33]

Table 2 presents the values of key macroeconomic variables in the BGP equilibrium

[33]This exercise can be rationalized with a tax/subsidy to the cost of recruiting inventors in a version of our model that includes an endogenous vacancy posting decision (e.g. Pijoan-Mas and Roldan-Blanco, 2025).

Table 2: Macroeconomic effects of frictions in the market for inventors

	(A)	(B)	(C)
	No mobility ($A \to 0$)	Baseline ($A = 0.1118$)	Full mobility ($A \to +\infty$)
Inventor mobility (A-to-U) rate	0.26%	8.58%	$+\infty$
Average inventor productivity	1.91	2.41	4.37
Probability of innovation by inventors	95.73%	45.56%	1.95%
Initial output (baseline = 1)	1.0235	1	0.9909
Consumption share	91.91%	91.39%	92.97%
Implementation spending share	1.07%	1.80%	2.51%
Research spending share	0.55%	0.34%	0.00%
Growth rate	**1.19%**	**2.00%**	**2.94%**

Notes: BGP equilibrium values for key variables in the baseline calibration (column (B)), a counterfactual economy with no inventor mobility (column (A)), and a counterfactual with full mobility (column (C)). *Variables:* Most variable calculations are explained in the main text. "Initial output" is expressed relative to the baseline (=1). The "Implementation rate" is the share of GDP accounted for by firm costs of implementing inventors' ideas. The "Research spending rate" is the share of GDP accounted for by (attached and unattached) inventor costs of generating ideas.

of three different scenarios. Column (B) is the baseline calibration. Column (A) is a counterfactual in which every parameter remains constant at its calibrated value, except for A, which is set to virtually zero. Column (C) shows the case where mobility is highest by letting A take on a very large number.[34]

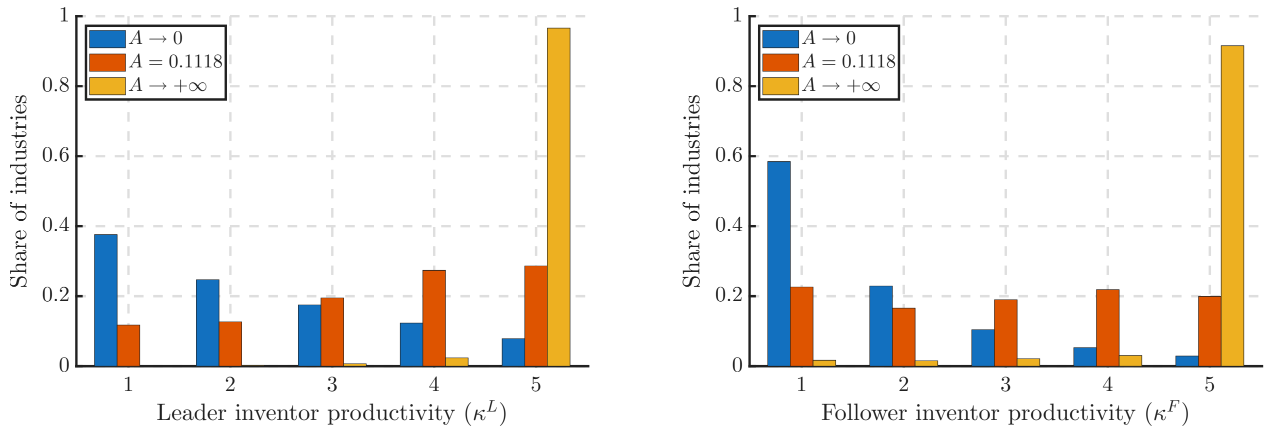
Our results show that, as efficiency in the market for inventors improves, the equilibrium distribution of inventors shifts towards better (larger-κ) types: average inventor productivity (the average κ in the economy) increases up toward the upper bound, set to $\overline{\kappa} = 5$. Figure 8 further illustrates this result by showing the distribution of industries by inventor type: as inventor mobility increases, this distribution shifts toward higher inventor productivity levels, both for inventors attached to leader firms (left plot) as well as those in follower firms (right plot). Therefore, as inventor mobility increases, misallocation between inventors and firms weakens, which allows the economy to reallocate toward more talented inventors.

Interestingly, as labor market frictions disappear and inventor mobility increases, the generation of knowledge gets increasingly concentrated: the probability that the average inventor, whether attached to a firm or unattached, produces a new idea drops sharply (as $A \to +\infty$, only 1.95% of inventors find ideas).[35] Thus, in equilibrium, there stock

[34]The structure of the model does not let us numerically solve for the equilibrium for the cases of $A = 0$ and $A = +\infty$, so in practice we set $A \approx 10^{-6}$ and $A \approx 400$, respectively.

[35]Indeed, as most inventors are at the upper bound $\overline{\kappa}$, they have no further incentive to keep innovating.

Figure 8: Equilibrium distribution of industries by inventor types attached to leader and follower firms.



Notes: This figure shows the distribution of industries by type of inventor attached to leader (left-hand panel) or follower (right-hand panel) firms, across our three counterfactual experiments.

of ideas gets replenished rarely. However, as innovations are increasingly produced by highly talented inventors, these innovations are of high quality and are implemented at a much higher frequency by firms.[36] In short, innovation becomes a rare but impactful event —one that firms seek to seize with much higher likelihood. As a result, the growth rate of the economy in the economy with full mobility is almost a full percentage point higher relative to the baseline. Or, in other words, frictions in the labor market for inventors are responsible for a 32% loss in economic growth.

The fact that inventors' productivity is larger and that innovations are on average more impactful also implies, as a by-product, that the technology gap distribution towards larger gaps, leading to a higher degree of concentration in the product market. This generates static misallocation which pushes down the initial output. Combined with the increase in the R&D share, the fall in initial output reduces the level of initial consumption.

5.2 Policy

Our previous analysis suggests that relaxing matching frictions among inventors should trigger powerful reallocation effects, creating incentives for higher-productivity inventors to thrive and for firms to increase their efforts to implement their radical new ideas. This opens up interesting new avenues for policy: beyond the well-known effects of R&D subsidies for the creation and dissemination of knowledge spillovers in the economy, could these type of industrial policies also have positive effects on growth and welfare by helping weaken

[36]Indeed, as shown in the table, research spending vanishes as a share of GDP (the terms related to z and z^U in equation (28)), but implementation spending (the terms related to x) soars.

misallocation forces between inventors and firms?

To study this question, we evaluate the effects of standard policies: (i) subsidies to inventors' research spending, and (ii) subsidies to firms' implementation spending. In each of these two experiments, we assume that a share τ_χ (respectively, τ_ξ) of the total research (respectively, implementation) costs incurred by the inventor (respectively, the firm) are paid directly by the government. The government, in turn, subsidizes this spending by levying lump-sum taxes to the representative consumer.[37]

5.2.1 Aggregate effects

Figure 9 shows the results of these simple experiments by plotting, on a grid of values for τ_χ and τ_ξ, various equilibrium variables computed from the BGP equilibrium using the same parameters as in our baseline calibration. The vertical dashed lines on each plot show the corresponding *optimal policies*, namely the rates τ_χ and τ_ξ that maximize consumption-equivalent welfare for the representative consumer. Welfare calculations are detailed in Appendix A.7.[38]

Subsidies to either inventors' research costs or to firms' implementation spending both increase inventor productivity substantially, making inventors more attractive for firms and, therefore, increasing the share of inventors that are attached to a firm. Subsidies, however, reduce inventor mobility, making relationships between firms and their existing inventors last longer. The reason for this is that subsidies increase the value of existing matches: the option of retaining inventors is more valuable under positive subsidies to either research or implementation because implementing innovations from in-house researchers becomes cheaper relative to spending costs on hiring new, more talented inventors. As firms rely more on the innovations from incumbent inventors than from new ones, inventors increase their innovation spending and find new ideas more frequently.

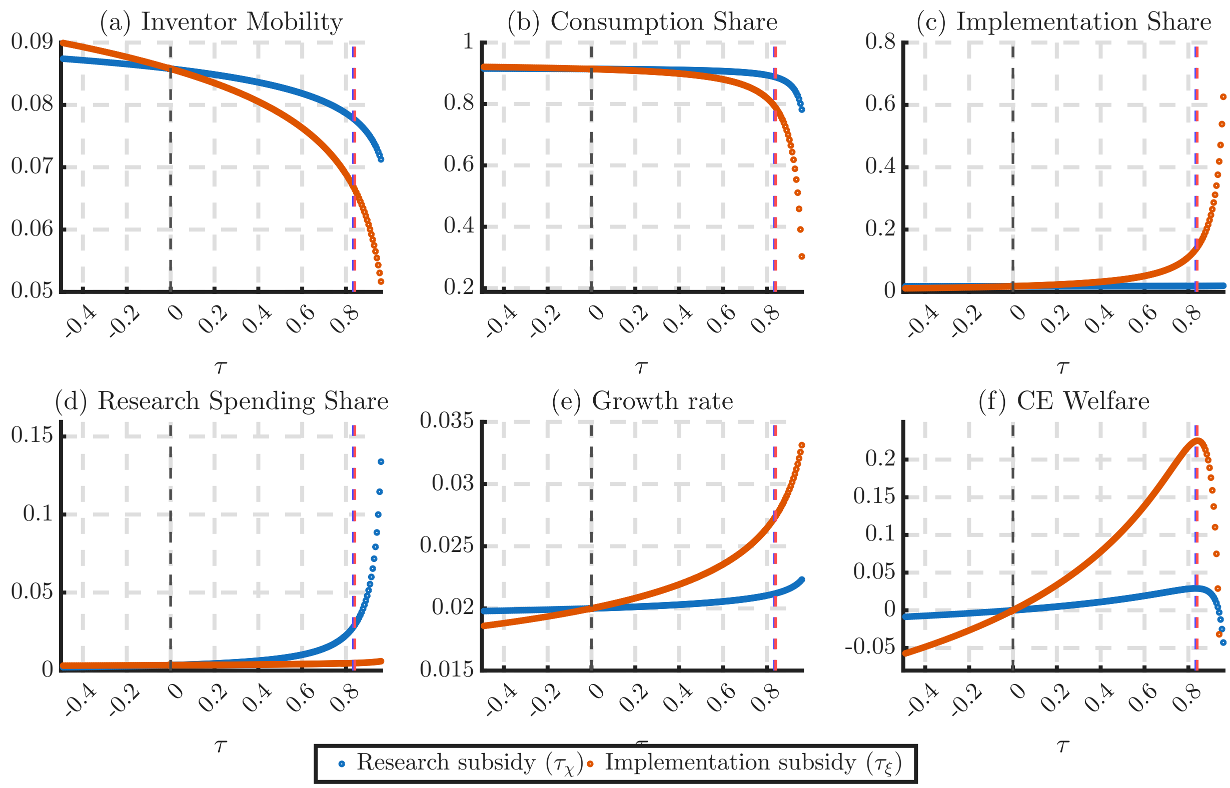
In terms of growth and welfare, subsidies are beneficial, albeit by different amounts. As usual in these models, subsidizing innovation increases growth but diverts resources away from consumption, which poses a trade-off for welfare. At the optimal policy of $\tau_\xi = 0.8402$ for subsidies to implementation (respectively, $\tau_\chi = 0.8339$ for subsidies to research), the growth rate of the economy is 0.12 percentage points (respectively, 0.74 percentage points) higher than in the baseline. Compared to the growth rate of 2.94% that we found in

[37] That is, the firm pays $(1-\tau_\xi)\xi x^\phi$ in implementation costs, and the inventor pays $(1-\tau_\chi)\chi z^\phi$ in research costs. When $\tau_\xi < 0$ and $\tau_\chi < 0$, the agents are being taxed, and the proceeds are still rebated lump-sum to the representative consumer.

[38] As we justify in this appendix, our notion of social welfare excludes inventor utilities and considers only the representative consumer.

Table 2 (column (C)) for the economy without labor market frictions, this implies that implementation subsidies are able to close about 80% of the missing gap in growth that is generated by search frictions, while research subsidies manage to close only about 13% of this gap. This is reflected in terms of welfare: under the optimal implementation subsidy, welfare gains in consumption-equivalent terms are 22.5% vs the much more modest 2.90% gain from the policy that subsidizes inventors' research directly.

Figure 9: Effects of research an implementation subsidies on BGP equilibrium variables.



Notes: This figure shows, on separate grids for τ_χ (in blue) and τ_ξ (in red), the BGP equilibrium values of selected aggregate variables. The black vertical line marks $\tau = 0$, and the red (respectively, blue) vertical lines denote the welfare-maximizing τ_ξ (respectively, τ_χ). Consumption-equivalent welfare is computed as described in Appendix A.7.

5.2.2 Distributional effects

Figures C.1 and C.2 in Appendix C show the distributional effects of these optimal policies. Figure C.1 reproduces Figure 7, where we showed the share of good (above median productivity) and bad (below median productivity) inventors that are allocated in concentrated (above-median HHI) versus competitive (below-median HHI) industries. We show this for the baseline calibration, the optimal research subsidy (τ_χ), and the optimal implementation subsidy (τ_χ). Similarly, Figure C.2 shows the equilibrium distribution of industries by inventor type for each of these three scenarios.

These figures show that, relative to the baseline, research subsidies (which, remember, are less effective in raising growth) increase both the share of good inventors in competitive industries, as well as the share of bad inventors in concentrated industries. That is, research subsidies make the sorting effects in the decentralized equilibrium (driving good inventors to firms with high implementation incentives) stronger. Figure C.2 then shows that this reallocation in sorting increases inventor productivity on average: the share of industries with high κ inventors increases relative to the no-policy benchmark.

Under implementation subsidies (which are far more effective in raising growth), the reallocation effects are quite different: while the share of good inventors in competitive industries remains roughly unchanged relative to the no-policy scenario, bad inventors are now disproportionately driven away from concentrated industries and toward competitive industries. As seen in Figure C.2, the implications for the distribution of industries are also quite different: in general, the share of industries with bad inventors increases relative to the baseline.

In sum, implementation subsidies more effectively increase growth and welfare, but they do so at the expense of a reallocation of industries away from talented inventors. Growth is higher not because inventors are better, but because firms implement their ideas a lot more frequently. Research subsidies, in contrast, foster talent: they induce firms to hire more talented inventors, so growth is higher not because firms implement more ideas, but because those ideas are better.

6 Conclusions

This paper studies the labor market for inventors to understand the importance of efficient inventor-firm matching in fostering aggregate innovation and economic growth. Empirical evidence from patent and firms' balance-sheet data reveals significant turnover in the inventor market. Firms that operate in highly competitive sectors have the highest hiring rates of inventors, but many high-productivity inventors are employed by dominant firms in concentrated industries.

Theoretically, we develop an endogenous growth model that incorporates a frictional labor market for inventors, where firms compete strategically to attract high-productivity inventors, and inventors improve their productivity by finding new ideas. We use the model to study the growth and welfare implications of the market for inventors. Our results indicate that improved matching efficiency in this market reallocates high-productivity inventors to firms with high R&D intensity, boosting aggregate innovation, growth and welfare.

Quantitatively, labor market frictions in the market for inventors yield losses in aggregate productivity growth of about 30%. Addressing frictions in the inventor market can thus lead to significant gains in R&D productivity, driving long-term economic growth and welfare. Indeed, when looking at R&D industrial policy, we find that under the optimal subsidies to research efforts (by inventors) and implementation efforts (by firms), economic growth increases between 0.12 and 0.74 percentage points, for overall gains in consumption-equivalent welfare that can be as high as 22.5%. Exploring more sophisticated policies, including combined policies and policies to firm entry, are interesting avenues for future research.

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Economic Growth when Knowledge is Concentrated

by Andrea Guccione and Pau Roldan-Blanco

Appendix Materials

A Theory Appendix

A.1 Inventor's and Firm's Value Functions

Let us define the short-hand notation for the job-filling rate in submarket (κ, E) by:

$$\zeta_\kappa(E) \equiv \eta \circ \mu^{-1}(\boldsymbol{\Phi}_\kappa(E)) \tag{A.1}$$

where the $\Phi_\kappa(E)$ mapping is defined in equation (18).

A.1.1 Inventor' Value

The value of an inventor of type κ attached to a firm with technology gap n, facing a competitor employing an inventor of type κ^-, under contract $\mathcal{C} \equiv \left(\widehat{w}, \{E'(n', \kappa', \kappa^{-\prime})\}\right)$ solves:

$$\begin{aligned}
r\boldsymbol{E}(n,\kappa,\kappa^-) - \partial_t \boldsymbol{E}(n,\kappa,\kappa^-) \;&=\; \widehat{w} \\
\textit{Inventor innovates} \quad & + \max_z \left\{ z\Big(E'\Big(n, \min(\kappa+1, \overline{\kappa}), \kappa^-\Big) - \boldsymbol{E}(n,\kappa,\kappa^-)\Big) - \chi z^{\phi} \boldsymbol{Y} \right\} \\
\textit{Knowledge capital depreciates} \quad & + \delta\Big(E'\Big(n, \max(\kappa-1, \underline{\kappa}), \max(\kappa^- - 1, \underline{\kappa})\Big) - \boldsymbol{E}(n,\kappa,\kappa^-)\Big) \\
\textit{Firm hires new inventor} \quad & + \sum_{\kappa' \in \mathbb{K}} \zeta_{\kappa'}\Big(E'(n,\kappa',\kappa^-)\Big)\Big(\boldsymbol{U}_\kappa - \boldsymbol{E}(n,\kappa,\kappa^-)\Big) \\
\textit{Competitor's inventor innovates} \quad & + \widetilde{z}\Big(E'\Big(n,\kappa,\min(\kappa^- + 1, \overline{\kappa})\Big) - \boldsymbol{E}(n,\kappa,\kappa^-)\Big) \\
\textit{Competitor hires new inventor} \quad & + \sum_{\kappa^{-\prime} \in \mathbb{K}} \zeta_{\kappa^{-\prime}}\Big(\widetilde{E}'(-n, \kappa^{-\prime}, \kappa)\Big)\Big(E'\big(n,\kappa,\kappa^{-\prime}\big) - \boldsymbol{E}(n,\kappa,\kappa^-)\Big) \\
\textit{Firm implements innovation} \quad & + x\Big(E'\big(\min(n+\kappa, \overline{n}), \kappa, \kappa^-\big) - \boldsymbol{E}(n,\kappa,\kappa^-)\Big) \\
\textit{Competitor implements innovation} \quad & + \widetilde{x}\Big(E'\big(\max(n-\kappa^-, -\overline{n}), \kappa, \kappa^-\big) - \boldsymbol{E}(n,\kappa,\kappa^-)\Big)
\end{aligned} \tag{A.2}$$

$$
\begin{aligned}
&\textit{Follower catches up to leader} && +\psi\Big(E'(0,\kappa,\kappa^-)-\boldsymbol{E}(n,\kappa,\kappa^-)\Big)\\
&\textit{Entry (firm is leader)} && +\mathbb{1}_{\{n>0\}}\sum_{\kappa'\in\mathbb{K}}\mathtt{m}^P\zeta_{\kappa'}\left(E^E_{\kappa'}(n,\kappa,\kappa^-)\right)\left(E'(n,\kappa,\kappa')-\boldsymbol{E}(n,\kappa,\kappa^-)\right)\\
&\textit{Entry (firm is follower)} && +\mathbb{1}_{\{n<0\}}\sum_{\kappa'\in\mathbb{K}}\mathtt{m}^P\zeta_{\kappa'}\left(E^E_{\kappa'}(-n,\kappa,\kappa^-)\right)\left(\boldsymbol{U}_\kappa-\boldsymbol{E}(n,\kappa,\kappa^-)\right)\\
&\textit{Entry (firms neck-to-neck)} && +\mathbb{1}_{\{n=0\}}\sum_{\kappa'\in\mathbb{K}}\mathtt{m}^P\zeta_{\kappa'}\left(E^E_{\kappa'}(n,\kappa,\kappa^-)\right)\left(\frac{1}{2}E'(n,\kappa,\kappa')+\frac{1}{2}\boldsymbol{U}_\kappa-\boldsymbol{E}(n,\kappa,\kappa^-)\right)
\end{aligned}
$$

The first-order condition for the inventor innovating while attached to a firm gives:

$$
z_{n,\kappa,\kappa^-}=\left(\frac{E'(n,\min(\kappa+1,\overline{\kappa}),\kappa^-)-\boldsymbol{E}(n,\kappa,\kappa^-)}{\chi\phi\boldsymbol{Y}}\right)^{\frac{1}{\phi-1}} \tag{A.3}
$$

A.1.2 Firm's Value

The value of a firm of type n employing an inventor of type $\kappa\in\mathbb{K}$, facing a competitor with inventor of type κ^-, under promised value R, solves the following HJB equation:[39]

$$
r\boldsymbol{V}(n,\kappa,\kappa^-,E)-\partial_t\boldsymbol{V}(n,\kappa,\kappa^-,E)=\max_{\substack{\{E(n',\kappa',\kappa^{-\prime})\}\\ \widehat{w},\ x}}\Big\{\pi(n)\boldsymbol{Y}-\widehat{w}-\xi x^\phi\boldsymbol{Y} \tag{A.4}
$$

$$
\begin{aligned}
&\textit{Firm's inventor innov.} && +z\left(\boldsymbol{V}\Big(n,\min(\kappa+1,\overline{\kappa}),\kappa^-,E'(n,\min(\kappa+1,\overline{\kappa}),\kappa^-)\Big)-\boldsymbol{V}(n,\kappa,\kappa^-,E)\right)\\
&\textit{Knowledge capital deprec.} && +\delta\bigg(\boldsymbol{V}\Big(n,\max(\kappa-1,\underline{\kappa}),\max(\kappa^--1,\underline{\kappa}),E'\left(n,\max(\kappa-1,\underline{\kappa}),\max(\kappa^--1,\underline{\kappa})\right)\Big)\dots\\
& && \cdots-\boldsymbol{V}(n,\kappa,\kappa^-,E)\bigg)\\
&\textit{Firm hires inventor} && +\sum_{\kappa'\in\mathbb{K}}\zeta_{\kappa'}\left(E'(n,\kappa',\kappa^-)\right)\left(\boldsymbol{V}\Big(n,\kappa',\kappa^-,E'(n,\kappa',\kappa^-)\Big)-\boldsymbol{V}(n,\kappa,\kappa^-,E)\right)\\
&\textit{Competitor's inventor innov.} && +\widetilde{z}\left(\boldsymbol{V}\Big(n,\kappa,\min(\kappa^-+1,\overline{\kappa}),E'(n,\kappa,\min(\kappa^-+1,\overline{\kappa})\Big)-\boldsymbol{V}(n,\kappa,\kappa^-,E)\right)\\
&\textit{Competitor hires inventor} && +\sum_{\kappa^{-\prime}\in\mathbb{K}}\zeta_{\kappa^{-\prime}}\left(\widetilde{E}'(-n,\kappa^{-\prime},\kappa)\right)\left(\boldsymbol{V}\Big(n,\kappa,\kappa^{-\prime},E'(n,\kappa,\kappa^{-\prime})\Big)-\boldsymbol{V}(n,\kappa,\kappa^-,E)\right)\\
&\textit{Firm implements} && +x\left(\boldsymbol{V}\Big(\min(n+\kappa,\overline{n}),\kappa,\kappa^-,E'(\min(n+\kappa,\overline{n}),\kappa,\kappa^-)\Big)-\boldsymbol{V}(n,\kappa,\kappa^-,E)\right)
\end{aligned}
$$

[39]Competitor's policies, both those of the other large incumbent and those of the potential entrant, which the firm takes as given, are marked with tildes.

$$
\begin{aligned}
&\textit{Competitor implements} && +\widetilde{x}\Big(\boldsymbol{V}\Big(\max(n-\kappa^-,-\overline{n}),\kappa,\kappa^-,E'(\max(n-\kappa^-,-\overline{n}),\kappa,\kappa^-)\Big)-\boldsymbol{V}(n,\kappa,\kappa^-,E)\Big)\\
&\textit{Follower catches up to leader} && +\psi\Big(\boldsymbol{V}\Big(0,\kappa,\kappa^-,E'(0,\kappa,\kappa^-)\Big)-\boldsymbol{V}(n,\kappa,\kappa^-,E)\Big)\\
&\textit{Entry (firm is leader)} && +\mathbb{1}_{\{n>0\}}\sum_{\kappa'\in\mathbb{K}}\mathtt{m}^P\zeta_{\kappa'}\left(\widetilde{E}^E_{\kappa'}(n,\kappa,\kappa^-)\right)\Big(\boldsymbol{V}(n,\kappa,\kappa',E'(n,\kappa,\kappa'))-\boldsymbol{V}(n,\kappa,\kappa^-,E)\Big)\\
&\textit{Entry (firm is follower)} && +\mathbb{1}_{\{n<0\}}\sum_{\kappa'\in\mathbb{K}}\mathtt{m}^P\zeta_{\kappa'}\left(\widetilde{E}^E_{\kappa'}(-n,\kappa,\kappa^-)\right)\Big(0-\boldsymbol{V}(n,\kappa,\kappa^-,E)\Big)\\
&\textit{Entry (firms neck-to-neck)} && +\mathbb{1}_{\{n=0\}}\sum_{\kappa'\in\mathbb{K}}\mathtt{m}^P\zeta_{\kappa'}\left(\widetilde{E}^E_{\kappa'}(n,\kappa,\kappa^-)\right)\Big(\frac{1}{2}\boldsymbol{V}(n,\kappa,\kappa',E'(n,\kappa,\kappa'))-\boldsymbol{V}(n,\kappa,\kappa^-,E)\Big)\Big\}
\end{aligned}
$$

The problem is subject to promise-keeping and inventor participation constraints:

$$
\begin{aligned}
\boldsymbol{E}(n,\kappa,\kappa^-) &\geq E\\
E'(n',\kappa',\kappa^{-\prime}) &\geq \boldsymbol{U}_{\kappa'} \quad \forall(n',\kappa',\kappa^{-\prime})
\end{aligned}
$$

A.2 Joint Surplus Equivalence

Denote by $\overline{\mathcal{P}}=\{\overline{w}_\kappa\boldsymbol{Y},\overline{e}'(n',\kappa',\kappa^{-\prime})\boldsymbol{Y},\overline{x}\}$ a generic policy of a firm of type n employing an inventor of type κ and facing a competitor with inventor κ^-. Further, denote by $\overline{z}$ a generic policy of the inventor attached to this firm. Then, the value of the firm, normalized by aggregate output, solves:

$$
\boldsymbol{v}(n,\kappa,\kappa^-,E)=\max_{\overline{\mathcal{P}},\,\overline{z}}\ \widehat{\boldsymbol{v}}\left(n,\kappa,\kappa^-|\overline{\mathcal{P}},\overline{z}\right)\quad \text{s.t. } \boldsymbol{e}(n,\kappa,\kappa^-)\geq e,\ \ e'(n',\kappa',\kappa^{-\prime})\geq \boldsymbol{u}_{\kappa'}\ \ \forall(n',\kappa',\kappa^{-\prime})
$$

where $e=E/\boldsymbol{Y}$ and $\widehat{\boldsymbol{v}}\left(n,\kappa,\kappa^-|\overline{\mathcal{P}},\overline{z}\right)$ satisfies

$$
\begin{aligned}
\rho(n,\kappa,\kappa^-)\widehat{\boldsymbol{v}}\left(n,\kappa,\kappa^-|\overline{\mathcal{P}},\overline{z}\right) &= \pi(n)-\overline{w}_\kappa-\xi\,\overline{x}^\phi+\overline{z}\boldsymbol{v}\left(n,\min(\kappa+1,\overline{\kappa}),\kappa^-,\overline{E}'(n,\min(\kappa+1,\overline{\kappa}),\kappa^-)\right)\\
&+\delta\boldsymbol{v}\Big(n,\max(\kappa-1,\underline{\kappa}),\max(\kappa^--1,\underline{\kappa}),\overline{E}'(n,\max(\kappa-1,\underline{\kappa}),\max(\kappa^--1,\underline{\kappa}))\Big)\\
&+\sum_{\kappa'\in\mathbb{K}}\zeta_{\kappa'}\left(\overline{e}'(n,\kappa',\kappa^-)\right)\boldsymbol{v}\left(n,\kappa',\kappa^-,\overline{E}'(n,\kappa',\kappa^-)\right)\\
&+\widetilde{z}\boldsymbol{v}\left(n,\kappa,\min(\kappa^-+1,\overline{\kappa}),\overline{E}'(n,\kappa,\min(\kappa^-+1,\overline{\kappa})\right)\\
&+\sum_{\kappa^{-\prime}\in\mathbb{K}}\zeta_{\kappa^{-\prime}}\left(\widetilde{e}'(-n,\kappa^{-\prime},\kappa)\right)\boldsymbol{v}\left(n,\kappa,\kappa^{-\prime},\overline{E}'(n,\kappa,\kappa^{-\prime})\right)\\
&+\overline{x}\boldsymbol{v}\left(\min(n+\kappa,\overline{n})_{,\kappa,\kappa^-},\overline{E}'(\min(n+\kappa,\overline{n}),\kappa,\kappa^-)\right)\\
&+\widetilde{x}\boldsymbol{v}\left(\max(n-\kappa^-,-\overline{n}),\kappa,\kappa^-,\overline{E}'(\max(n-\kappa^-,-\overline{n}),\kappa,\kappa^-)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \psi \boldsymbol{v}\left(0,\kappa,\kappa^-,\overline{E}'(0,\kappa,\kappa^-)\right) \\
&+ \mathbb{1}_{\{n>0\}} \sum_{\kappa'\in\mathbb{K}} \mathtt{m}^P \zeta_{\kappa'}\left(\widetilde{e}^E_{\kappa'}(n,\kappa,\kappa^-)\right) \boldsymbol{v}\left(n,\kappa,\kappa',\overline{E}'(n,\kappa,\kappa')\right) \\
&+ \mathbb{1}_{\{n=0\}} \sum_{\kappa'\in\mathbb{K}} \mathtt{m}^P \zeta_{\kappa'}\left(\widetilde{e}^E_{\kappa'}(n,\kappa,\kappa^-)\right) \frac{1}{2}\boldsymbol{v}\left(n,\kappa,\kappa',\overline{E}'(n,\kappa,\kappa')\right)
\end{aligned} \tag{A.5}
$$

with

$$
\begin{aligned}
\rho(n,\kappa,\kappa^-) \ \equiv\ & \rho + \overline{z} + \delta + \sum_{\kappa'\in\mathbb{K}} \zeta_{\kappa'}\left(\overline{e}'(n,\kappa',\kappa^-)\right) + \widetilde{z} + \sum_{\kappa^{-\prime}\in\mathbb{K}} \zeta_{\kappa^{-\prime}}\left(\widetilde{e}'(-n,\kappa^{-\prime},\kappa)\right) + \overline{x} + \widetilde{x} + \psi \\
& + \mathbb{1}_{\{n\geq 0\}} \sum_{\kappa'\in\mathbb{K}} \mathtt{m}^P \zeta_{\kappa'}\left(\widetilde{e}^E_{\kappa'}(n,\kappa,\kappa^-)\right) + \mathbb{1}_{\{n<0\}} \sum_{\kappa'\in\mathbb{K}} \mathtt{m}^P \zeta_{\kappa'}\left(\widetilde{e}^E_{\kappa'}(-n,\kappa,\kappa^-)\right).
\end{aligned}
$$

Once again, competitor policies are marked with tildes, and taken as given by the firm.

Under policies $\overline{\mathcal{P}}$ and $\overline{z}$, the value of the inventor, normalized by aggregate output, solves:

$$
\begin{aligned}
\rho(n,\kappa,\kappa^-)\boldsymbol{e}\left(n,\kappa,\kappa^-|\overline{\mathcal{P}},\overline{z}\right) \ =\ & \overline{w}_\kappa - \chi\,\overline{z}^\phi + \overline{z}\,\overline{e}'\left(n,\min(\kappa+1,\overline{\kappa}),\kappa^-\right) \\
& + \delta\overline{e}'\left(n,\max(\kappa-1,\underline{\kappa}),\max(\kappa^- - 1,\underline{\kappa})\right) \\
& + \sum_{\kappa'\in\mathbb{K}} \zeta_{\kappa'}\left(\overline{e}'(n,\kappa',\kappa^-)\right)\boldsymbol{u}_\kappa + \widetilde{z}\,\overline{e}'\left(n,\kappa,\min(\kappa^-+1,\overline{\kappa})\right) \\
& + \sum_{\kappa^{-\prime}\in\mathbb{K}} \zeta_{\kappa^{-\prime}}\left(\widetilde{e}'(-n,\kappa^{-\prime},\kappa)\right)\overline{e}'\left(n,\kappa,\kappa^{-\prime}\right) + \overline{x}\,\overline{e}'\left(\min(n+\kappa,\overline{n}),\kappa,\kappa^-\right) \\
& + \widetilde{x}\,\overline{e}'\left(\max(n-\kappa^-,-\overline{n}),\kappa,\kappa^-\right) + \psi\overline{e}'(0,\kappa,\kappa^-) \\
& + \mathbb{1}_{\{n>0\}} \sum_{\kappa'\in\mathbb{K}} \mathtt{m}^P \zeta_{\kappa'}\left(\widetilde{e}^E_{\kappa'}(n,\kappa,\kappa^-)\right)\overline{e}'(n,\kappa,\kappa') \\
& + \mathbb{1}_{\{n<0\}} \sum_{\kappa'\in\mathbb{K}} \mathtt{m}^P \zeta_{\kappa'}\left(\widetilde{e}^E_{\kappa'}(-n,\kappa,\kappa^-)\right)\boldsymbol{u}_\kappa \\
& + \mathbb{1}_{\{n=0\}} \sum_{\kappa'\in\mathbb{K}} \mathtt{m}^P \zeta_{\kappa'}\left(\widetilde{e}^E_{\kappa'}(n,\kappa,\kappa^-)\right)\left(\frac{1}{2}\overline{e}'(n,\kappa,\kappa') + \frac{1}{2}\boldsymbol{u}_\kappa\right)
\end{aligned} \tag{A.6}
$$

In equilibrium, the promise-keeping constraint must hold with equality. From equation (A.6) we can then solve for the wage $\overline{w}_\kappa$ such that $\boldsymbol{e}(n,\kappa,\kappa^-|\overline{\mathcal{P}},\overline{z}) = e$:

$$
\begin{aligned}
\overline{w}_\kappa \ =\ & \rho(n,\kappa,\kappa^-)e + \chi\,\overline{z}^\phi - \overline{z}\,\overline{e}'\left(n,\min(\kappa+1,\overline{\kappa}),\kappa^-\right) \\
& - \delta\overline{e}'\left(n,\max(\kappa-1,\underline{\kappa}),\max(\kappa^- - 1,\underline{\kappa})\right)
\end{aligned} \tag{A.7}
$$

$$
\begin{aligned}
&- \sum_{\kappa' \in \mathbb{K}} \zeta_{\kappa'} \Big(\overline{e}'(n,\kappa',\kappa^-) \Big) \boldsymbol{u}_\kappa - \widetilde{z}\, \overline{e}' \Big(n,\kappa,\min(\kappa^- + 1, \overline{\kappa}) \Big) \\
&- \sum_{\kappa^{-\prime} \in \mathbb{K}} \zeta_{\kappa^{-\prime}} \Big(\widetilde{e}'(-n,\kappa^{-\prime},\kappa) \Big) \overline{e}' \Big(n,\kappa,\kappa^{-\prime} \Big) - \overline{x}\, \overline{e}' \Big(\min(n+\kappa, \overline{n}), \kappa, \kappa^- \Big) \\
&- \widetilde{x}\, \overline{e}' \big(\max(n-\kappa^-, -\overline{n}), \kappa, \kappa^- \big) - \psi \overline{e}'(0,\kappa,\kappa^-) \\
&- \mathbb{1}_{\{n>0\}} \sum_{\kappa' \in \mathbb{K}} \mathtt{m}^P \zeta_{\kappa'} \left(\widetilde{e}^E_{\kappa'}(n,\kappa,\kappa^-) \right) \overline{e}'(n,\kappa,\kappa') - \mathbb{1}_{\{n<0\}} \sum_{\kappa' \in \mathbb{K}} \mathtt{m}^P \zeta_{\kappa'} \left(\widetilde{e}^E_{\kappa'}(-n,\kappa,\kappa^-) \right) \boldsymbol{u}_\kappa \\
&- \mathbb{1}_{\{n=0\}} \sum_{\kappa' \in \mathbb{K}} \mathtt{m}^P \zeta_{\kappa'} \left(\widetilde{e}^E_{\kappa'}(n,\kappa,\kappa^-) \right) \left(\frac{1}{2} \overline{e}'(n,\kappa,\kappa') + \frac{1}{2} \boldsymbol{u}_\kappa \right)
\end{aligned}
$$

Plugging (A.7) into (A.5) we get:

$$
\begin{aligned}
\rho(n,\kappa,\kappa^-) &\Bigg(\underbrace{\widehat{v}\,(n,\kappa,\kappa^- | \overline{\mathcal{P}}, \overline{z}) + e}_{=\boldsymbol{\omega}(n,\kappa,\kappa^-,E|\overline{\mathcal{P}},\overline{z})} \Bigg) = \pi(n) - \xi\, \overline{x}^\phi - \chi\, \overline{z}^\phi \\
&+ \overline{z} \Bigg(\underbrace{\boldsymbol{v}\left(n, \min(\kappa+1,\overline{\kappa}), \kappa^-, \overline{E}'(n,\min(\kappa+1,\overline{\kappa}),\kappa^-)\right) + \overline{e}'\Big(n,\min(\kappa+1,\overline{\kappa}),\kappa^-\Big)}_{=\boldsymbol{\omega}(n,\min(\kappa+1,\overline{\kappa}),\kappa^-,\overline{E}'(n,\min(\kappa+1,\overline{\kappa})))} \Bigg) \\
&+ \delta \Bigg(\underbrace{\boldsymbol{v}\left(n, \max(\kappa-1,\underline{\kappa}), \max(\kappa^- - 1,\underline{\kappa}), \overline{e}'\Big(n,\max(\kappa-1,\underline{\kappa}),\max(\kappa^- - 1,\underline{\kappa})\Big)\right) \ldots}\\
&\qquad \underbrace{\cdots + \overline{e}'\Big(n,\max(\kappa-1,\underline{\kappa}),\max(\kappa^- - 1,\underline{\kappa})\Big)}_{=\boldsymbol{\omega}(n,\max(\kappa-1,\underline{\kappa}),\max(\kappa^- -1,\underline{\kappa}),\overline{E}'_\kappa(n,\max(\kappa-1,\underline{\kappa}),\max(\kappa^- -1,\underline{\kappa}))} \Bigg) \\
&+ \sum_{\kappa' \in \mathbb{K}} \zeta_{\kappa'} \Big(\overline{e}'(n,\kappa',\kappa^-) \Big) \Big(\underbrace{\boldsymbol{v}\Big(n,\kappa',\kappa^-,\overline{E}'(n,\kappa',\kappa^-)\Big)}_{=\boldsymbol{\omega}(n,\kappa',\kappa^-,\overline{E}'(n,\kappa',\kappa^-))-\overline{e}'(n,\kappa',\kappa^-)} + \boldsymbol{u}_\kappa \Big) \\
&+ \widetilde{z} \Bigg(\underbrace{\boldsymbol{v}\Big(n,\kappa,\min(\kappa^- + 1,\overline{\kappa}), \overline{E}'(n,\kappa,\min(\kappa^- + 1,\overline{\kappa})\Big) + \overline{e}'\Big(n,\kappa,\min(\kappa^- + 1,\overline{\kappa})\Big)}_{=\boldsymbol{\omega}(n,\kappa,\min(\kappa^- +1,\overline{\kappa}),\overline{E}'(n,\kappa,\min(\kappa^- +1,\overline{\kappa})))} \Bigg) \\
&+ \sum_{\kappa^{-\prime} \in \mathbb{K}} \zeta_{\kappa^{-\prime}} \Big(\widetilde{e}'(-n,\kappa^{-\prime},\kappa) \Big) \Bigg(\underbrace{\boldsymbol{v}\Big(n,\kappa,\kappa^{-\prime},\overline{E}'(n,\kappa,\kappa^{-\prime})\Big) + \overline{e}'\Big(n,\kappa,\kappa^{-\prime}\Big)}_{=\boldsymbol{\omega}(n,\kappa,\kappa^{-\prime},\overline{E}'(n,\kappa,\kappa^{-\prime}))} \Bigg) \\
&+ \overline{x} \Bigg(\underbrace{\boldsymbol{v}\Big(n+\kappa,\kappa,\kappa^-,\overline{E}'(n+\kappa,\kappa,\kappa^-)\Big) + \overline{e}'\Big(n+\kappa,\kappa,\kappa^-\Big)}_{=\boldsymbol{\omega}(n+\kappa,\kappa,\kappa^-,\overline{E}'(n+\kappa,\kappa,\kappa^-))} \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+\widetilde{x}\Bigg(\underbrace{\boldsymbol{v}\Big(n-\kappa^-,\kappa,\kappa^-,\overline{E}'(n-\kappa^-,\kappa,\kappa^-)\Big)+\overline{e}'(n-\kappa^-,\kappa,\kappa^-)}_{=\boldsymbol{\omega}(n-\kappa^-,\kappa,\kappa^-,\overline{E}'(n-\kappa^-,\kappa,\kappa^-))}\Bigg)\\
&+\psi\Bigg(\underbrace{\boldsymbol{v}\Big(0,\kappa,\kappa^-,\overline{E}'(0,\kappa,\kappa^-)\Big)+\overline{e}'(0,\kappa,\kappa^-)}_{=\boldsymbol{\omega}(0,\kappa,\kappa^-,\overline{E}'(0,\kappa,\kappa^-))}\Bigg)\\
&+\mathbb{1}_{\{n>0\}}\sum_{\kappa'\in\mathbb{K}}\mathtt{m}^P\zeta_{\kappa'}\left(e^E_{\kappa'}(n,\kappa,\kappa^-)\right)\Bigg(\underbrace{\boldsymbol{v}\left(n,\kappa,\kappa',\overline{E}'(n,\kappa,\kappa')\right)+\overline{e}'(n,\kappa,\kappa')}_{=\boldsymbol{\omega}\left(n,\kappa,\kappa',\overline{E}'(n,\kappa,\kappa')\right)}\Bigg)\\
&+\mathbb{1}_{\{n<0\}}\sum_{\kappa'\in\mathbb{K}}\mathtt{m}^P\zeta_{\kappa'}\left(e^E_{\kappa'}(-n,\kappa,\kappa^-)\right)\boldsymbol{u}_\kappa\\
&+\mathbb{1}_{\{n=0\}}\sum_{\kappa'\in\mathbb{K}}\mathtt{m}^P\zeta_{\kappa'}\left(e^E_{\kappa'}(n,\kappa,\kappa^-)\right)\Bigg(\frac{1}{2}\Big(\underbrace{\boldsymbol{v}\left(n,\kappa,\kappa',\overline{E}'(n,\kappa,\kappa')\right)+\overline{e}'(n,\kappa,\kappa')}_{=\boldsymbol{\omega}\left(n,\kappa,\kappa',\overline{E}'(n,\kappa,\kappa')\right)}\Big)+\frac{1}{2}\boldsymbol{u}_\kappa\Bigg)
\end{aligned}
\tag{A.8}
$$

where $\boldsymbol{\omega}\left(n,\kappa,\kappa^-,E|\overline{\mathcal{P}},\overline{z}\right)$ is the *normalized* joint surplus under policies $\overline{\mathcal{P}}$ and $\overline{z}$.

The right-hand side of equation (A.8) is independent of E. Hence, we can write:

$$
\begin{aligned}
\rho(n,\kappa,\kappa^-)\boldsymbol{\omega}\left(n,\kappa,\kappa^-|\overline{\mathcal{P}},\overline{z}\right)=\;&\pi(n)-\xi\,\overline{x}^\phi-\chi\,\overline{z}^\phi\\
&+\overline{z}\boldsymbol{\omega}(n,\min(\kappa+1,\overline{\kappa}),\kappa^-)+\delta\boldsymbol{\omega}(n,\max(\kappa-1,\underline{\kappa}),\max(\kappa^--1,\underline{\kappa}))\\
&+\sum_{\kappa'\in\mathbb{K}}\zeta_{\kappa'}\Big(\overline{e}'(n,\kappa',\kappa^-)\Big)\Big(\boldsymbol{\omega}(n,\kappa',\kappa^-)-\overline{e}'(n,\kappa',\kappa^-)+\boldsymbol{u}_\kappa\Big)\\
&+\widetilde{z}\boldsymbol{\omega}(n,\kappa,\min(\kappa^-+1,\overline{\kappa}))+\sum_{\kappa^{-\prime}\in\mathbb{K}}\zeta_{\kappa^{-\prime}}\Big(\widetilde{e}'(-n,\kappa^{-\prime},\kappa)\Big)\boldsymbol{\omega}(n,\kappa,\kappa^{-\prime})\\
&+\overline{x}\boldsymbol{\omega}(n+\kappa,\kappa,\kappa^-)+\widetilde{x}\boldsymbol{\omega}(n-\kappa^-,\kappa,\kappa^-)+\psi\boldsymbol{\omega}(0,\kappa,\kappa^-)\\
&+\mathbb{1}_{\{n>0\}}\sum_{\kappa'\in\mathbb{K}}\mathtt{m}^P\zeta_{\kappa'}\left(e^E_{\kappa'}(n,\kappa,\kappa^-)\right)\boldsymbol{\omega}(n,\kappa,\kappa')\\
&+\mathbb{1}_{\{n<0\}}\sum_{\kappa'\in\mathbb{K}}\mathtt{m}^P\zeta_{\kappa'}\left(e^E_{\kappa'}(-n,\kappa,\kappa^-)\right)\boldsymbol{u}_\kappa\\
&+\mathbb{1}_{\{n=0\}}\sum_{\kappa'\in\mathbb{K}}\mathtt{m}^P\zeta_{\kappa'}\left(e^E_{\kappa'}(n,\kappa,\kappa^-)\right)\left(\frac{1}{2}\boldsymbol{\omega}(n,\kappa,\kappa')+\frac{1}{2}\boldsymbol{u}_\kappa\right)
\end{aligned}
\tag{A.9}
$$

The maximized joint surplus is:

$$
\boldsymbol{\omega}\left(n,\kappa,\kappa^-\right)=\max_{\overline{\mathcal{P}},\overline{z}}\Big\{\boldsymbol{\omega}\left(n,\kappa,\kappa^-|\overline{\mathcal{P}},\overline{z}\right),\ \text{s.t. } e'(n',\kappa',\kappa^{-\prime})\geq\boldsymbol{u}_{\kappa'},\ \ \forall(n',\kappa',\kappa^{-\prime})\Big\}
$$

Given (A.9), rearranging terms, it is then easy to see that the joint surplus solves equation (21).

A.3 Equilibrium measures

A.3.1 Distribution over industry states

Let $\varphi_{m,\kappa^L,\kappa^F,t}$ be the measure of industries where, at time t, the leader is $m \in \{0, 1, 2, \ldots, \overline{n}\}$ steps ahead, employs an inventor of type $\kappa^L \in \mathbb{K}$ and the follower employs an inventor $\kappa^F \in \mathbb{K}$. The evolution of this measure is described by the differential equation:

$$\begin{aligned}
\partial_t \varphi_{m,\kappa^L,\kappa^F,t} = {} & \varphi_{m-\kappa^L,\kappa^L,\kappa^F,t}\, x_{(m-\kappa^L),\kappa^L,\kappa^F,t} \;+\; \varphi_{m,\kappa^L-1,\kappa^F,t}\, z_{m,\kappa^L-1,\kappa^F,t} \\
& + \sum_{\kappa' \in \mathbb{K}} \varphi_{m,\kappa',\kappa^F,t}\, \zeta_{\kappa^L} \left(e'_{m,\kappa',\kappa^F}(m, \kappa^L, \kappa^F) \right) \;+\; \varphi_{m+\kappa^F,\kappa^L,\kappa^F,t}\, x_{-(m+\kappa^F),\kappa^F,\kappa^L,t} \\
& + \varphi_{\kappa^F-m,\kappa^L,\kappa^F,t}\, x_{-(\kappa^F-m),\kappa^F,\kappa^L,t} \;+\; \varphi_{m,\kappa^L,\kappa^F-1,t}\, z_{-m,\kappa^F-1,\kappa^L,t} \\
& + \sum_{\kappa' \in \mathbb{K}} \varphi_{m,\kappa^L,\kappa',t}\, \zeta_{\kappa^F} \left(e'_{-m,\kappa',\kappa^L}(-m, \kappa^F, \kappa^L) \right) \;+\; \sum_{\kappa' \in \mathbb{K}} \varphi_{m,\kappa^L,\kappa',t}\, \mathtt{m}^P \zeta_{\kappa^F}(e^E_{\kappa^F}(m, \kappa^L, \kappa')) \\
& - \varphi_{m,\kappa^L,\kappa^F,t} \Bigg[x_{m,\kappa^L,\kappa^F,t} + x_{-m,\kappa^F,\kappa^L,t} + z_{m,\kappa^L,\kappa^F,t} + z_{-m,\kappa^F,\kappa^L,t} \\
& \qquad + \sum_{\kappa' \in \mathbb{K}} \zeta_{\kappa'} \left(e'_{m,\kappa^L,\kappa^F}(m, \kappa', \kappa^F) \right) + \sum_{\kappa' \in \mathbb{K}} \zeta_{\kappa'} \left(e'_{-m,\kappa^F,\kappa^L}(-m, \kappa', \kappa^L) \right) \\
& \qquad + \sum_{\kappa' \in \mathbb{K}} \mathtt{m}^P \zeta_{\kappa'} \left(e^E_{\kappa'}(m, \kappa^L, \kappa^F) \right) \Bigg]
\end{aligned} \tag{A.10}$$

The terms on the right-hand side represent, respectively: additions due to implemented innovations of leaders in $(m - \kappa^L, \kappa^L, \kappa^F)$; additions due to innovations of leaders' inventors in $(m, \kappa^L - 1, \kappa^F)$; additions due to leaders in (m, κ', κ^F) hiring inventors of type κ^L, $\forall \kappa'$; additions due to implemented innovations of followers in $(m + \kappa^F, \kappa^L, \kappa^F)$; additions due to innovations of followers' inventors in $(m, \kappa^L, \kappa^F - 1)$; additions due to followers in (m, κ^L, κ') hiring inventors of type κ^F, $\forall \kappa'$; additions due to entry in industries (m, κ^L, κ') with inventor κ^F, $\forall \kappa'$; subtractions due to innovation and hiring decisions of leaders and followers in (m, κ^L, κ^F); subtractions due to entry in (m, κ^L, κ^F).[40]

In the BGP equilibrium, we impose $\partial_t \varphi_{m,\kappa^L,\kappa^F,t} = 0$, $\forall (m, \kappa^L, \kappa^F)$, and solve for the resulting system of equations.

[40]Note that in equation (A.10): the first line is absent whenever $m - \kappa^L < 0$; the second line is absent if $\kappa^L = \underline{\kappa}$; the fourth is absent whenever $m + \kappa^F > \overline{n}$; the fifth is present only if $(\kappa^F - m \geq 0) \wedge (\kappa^L = \kappa^F)$; and the sixth is absent if $\kappa^F = \underline{\kappa}$.

A.3.2 Distribution of unattached inventors

Finally, the following flow equation pins down the stationary measure of unattached inventors of type κ, φ^U_κ, for each $\kappa \in \mathbb{K}$:

$$
\begin{aligned}
\partial_t \varphi^U_{\kappa,t} &= \varphi^U_{\kappa-1,t} z^U_{\kappa-1,t} + \varphi^U_{\kappa+1,t}\delta \\
&+ \sum_{n=-\bar{n}}^{\bar{n}} \sum_{\kappa^- \in \mathbb{K}} \varphi^E_{n,\kappa,\kappa^-} \left(\sum_{\kappa' \in \mathbb{K}} \zeta_{\kappa'}(e'_\kappa(n,\kappa',\kappa^-)) + \mathbb{1}_{\{n<0\}} \mathtt{m}^P \sum_{\kappa' \in \mathbb{K}} \zeta_{\kappa'}(e^E_{\kappa'}(-n,\kappa,\kappa^-)) \right) \\
&- \varphi^U_{\kappa,t} \left(z^U_{\kappa,t} + \delta + \sum_{m=0}^{\bar{n}} \sum_{\kappa^L \in \mathbb{K}} \sum_{\kappa^F \in \mathbb{K}} \varphi_{m,\kappa^L,\kappa^F} \left(\mu(\theta_\kappa(e'_{m,\kappa^L,\kappa^F}(m,\kappa,\kappa^F))) + \mu(\theta_\kappa(e'_{-m,\kappa^F,\kappa^L}(-m,\kappa,\kappa^L))) \right) \right)
\end{aligned}
\tag{A.11}
$$

where $\varphi^E_{n,\kappa,\kappa^-}$ is the measure of inventors of type κ attached to firms of type n facing a competitor with inventor of type κ^-, that is:[41]

$$
\varphi^E_{n,\kappa,\kappa^-} \equiv \begin{cases} \varphi_{n,\kappa,\kappa^-} & \text{if } n > 0 \\ \varphi_{-n,\kappa,\kappa^-} & \text{if } n < 0 \\ 2\varphi_{0,\kappa,\kappa^-} & \text{if } n = 0 \end{cases}
\tag{A.12}
$$

The terms on the right-hand side of (A.11) represent: additions due to innovations of unattached inventors of type $\kappa - 1$; additions due to knowledge capital depreciation of unattached inventors of type $\kappa + 1$; additions due to attached-to-unattached transitions; subtractions due to innovation, knowledge capital depreciation and transitions to employment of unattached inventors of type κ.

In the BGP equilibrium, we impose $\partial_t \varphi^U_{\kappa,t} = 0$, $\forall \kappa \in \mathbb{K}$, and solve for the resulting system of equations.

A.4 Growth

Let $q^L_{jt} \equiv \max\{q_{ijt}, q_{-ijt}, q_{cjt}\}$ denote the productivity of industry j's leader, and $\boldsymbol{Q}_t \equiv \exp\left(\int_0^1 \ln q^L_{jt} dj\right)$ be the productivity frontier of the economy. From the definition of the aggregate price index, we have

$$
\boldsymbol{P}_t = \exp\left[\frac{\sigma-1}{\sigma} \int_0^1 \ln\left(\sum_{f=i,-i,c} p_{fjt}^{\frac{\sigma}{\sigma-1}}\right) dj\right].
\tag{A.13}
$$

[41] As each firm employs one inventor, the measure of inventors equals the measure of firms, state by state. Moreover, in every industry there are two firms. Therefore, we can write the measure of attached inventors in terms of the distribution of industry states, as shown in (A.12).

Given the normalization $\boldsymbol{P}_t = 1$, and using $p_{fjt} = M_{fjt}\frac{w_t^P}{q_{fjt}}$, we can find an expression of the wage for production labor in equilibrium:

$$\frac{w_t^P}{\boldsymbol{Q}_t} = \exp\left[\frac{1-\sigma}{\sigma}\int_0^1 \ln\left(\sum_{f=i,-i,c}\left(\frac{q_{fjt}/q_{jt}^L}{M_{fjt}}\right)^{\frac{\sigma}{1-\sigma}}\right)dj\right]$$
$$= \exp\left[\frac{1-\sigma}{\sigma}\int_0^1 \ln\left(\left(\frac{\sigma(1-s_{ijt})}{1-\sigma s_{ijt}}\right)^{\frac{\sigma}{1-\sigma}} + \left(\lambda^{-m_{jt}}\frac{\sigma(1-s_{-ijt})}{1-\sigma s_{-ijt}}\right)^{\frac{\sigma}{1-\sigma}} + \alpha^{\frac{\sigma}{1-\sigma}}\right)dj\right], \tag{A.14}$$

where, without loss of generality, we have assumed that firm i is the leader and $-i$ is the follower. Aggregate output, $\boldsymbol{Y}_t$, is:

$$\ln(\boldsymbol{Y}_t) = \frac{1}{\sigma}\int_0^1 \ln\left(\sum_{f=i,-i,c} y_{fjt}^{\sigma}\right)dj \tag{A.15}$$
$$= \int_0^1 \ln\left(y_{ijt}\right)dj + \frac{1}{\sigma}\int_0^1 \ln\left(\sum_{f=i,-i,c}\left(\frac{y_{fjt}}{y_{ijt}}\right)^{\sigma}\right)dj$$
$$= \ln(\boldsymbol{Q}_t) + \underbrace{\int_0^1 \ln\left(s_{ijt}\left(\frac{M_{ijt}}{\boldsymbol{M}_t}\right)^{-1}\right)dj}_{\equiv \boldsymbol{\Psi}_t^A}$$
$$+ \underbrace{\frac{1}{\sigma}\int_0^1 \ln\left(1+\left(\lambda^{-m_{jt}}\frac{s_{-ijt}}{s_{ijt}}\frac{M_{ijt}}{M_{-ijt}}\right)^{\sigma} + \left(\alpha\frac{s_{cjt}}{s_{ijt}}M_{ijt}\right)^{\sigma}\right)dj}_{\equiv \boldsymbol{\Psi}_t^B}.$$

On the BGP, we can write:

$$\boldsymbol{\Psi}_t^A = \sum_{m=0}^{\bar{n}} \widehat{\varphi}_m \ln\left(\frac{\sigma(1-s_m^L)s_m^L}{1-\sigma s_m^L}\boldsymbol{M}_t\right), \tag{A.16}$$
$$\boldsymbol{\Psi}_t^B = \sum_{m=0}^{\bar{n}} \widehat{\varphi}_m \ln\left(1+\left(\lambda^{-m}\frac{s_m^F}{s_m^L}\frac{1-\sigma s_m^L}{1-s_m^L}\frac{1-s_m^F}{1-\sigma s_m^F}\right)^{\sigma} + \left(\alpha\frac{s_m^C}{s_m^L}\frac{1-\sigma s_m^L}{\sigma(1-s_m^L)}\right)^{\sigma}\right), \tag{A.17}$$

where the superscripts L, F and C mean leader, follower, and competitive fringe, respectively, and we have defined the marginal density:

$$\widehat{\varphi}_m \equiv \sum_{\kappa^L\in\mathbb{K}}\sum_{\kappa^F\in\mathbb{K}} \varphi_{m,\kappa^L,\kappa^F}. \tag{A.18}$$

In a BGP, the distribution of industries is stationary, so market shares are time-invariant. Thus, by equation (A.14) the normalized wage for production labor, w_t^P / Q_t, is constant. Similarly, by equation (27), the aggregate markup M_t is constant, so both Ψ_t^A and Ψ_t^B are constant. Therefore, by equation (A.15), $g \equiv \partial_t Y_t / Y_t = \partial_t Q_t / Q_t$, and by the resource constraint (equation (28)), $g = \partial_t C_t / C_t = r - \rho$.

In turn, the evolution of Q_t can be characterized as follows:

- In an industry where the leader is m steps ahead and employs an inventor of type κ^L, and where the follower employs an inventor of type $\kappa^F > m$, we have:

$$\ln(q_{jt+\Delta t}^L) = \begin{cases} \ln(\lambda^{\kappa^L} q_{jt}^L) & \text{w/p.} \quad x_{m,\kappa^L,\kappa^F} \Delta t + o(\Delta t) \\ \ln(\lambda^{\kappa^F - m} q_{jt}^L) & \text{w/p.} \quad x_{-m,\kappa^F,\kappa^L} \Delta t + o(\Delta t) \\ \ln(q_{jt}^L) & \text{w/p.} \quad 1 - (x_{m,\kappa^L,\kappa^F} + x_{-m,\kappa^F,\kappa^L}) \Delta t + o(\Delta t) \end{cases}$$

- In an industry where the leader is m steps ahead and employs an inventor of type κ^L, and where the follower employs an inventor of type $\kappa^F \leq m$, we have:

$$\ln(q_{jt+\Delta t}^L) = \begin{cases} \ln(\lambda^{\kappa^L} q_{jt}^L) & \text{w/p.} \quad x_{m,\kappa^L,\kappa^F} \Delta t + o(\Delta t) \\ \ln(q_{jt}^L) & \text{w/p.} \quad 1 - x_{m,\kappa^L,\kappa^F} \Delta t + o(\Delta t) \end{cases}$$

for an infinitesimal $\Delta t > 0$. Then, the growth rate of the productivity frontier is:

$$\begin{aligned} g = \frac{\partial_t Q_t}{Q_t} &= \lim_{\Delta t \to 0} \frac{\ln(Q_{t+\Delta t}) - \ln(Q_t)}{\Delta t} = \\ &= \sum_{m=0}^{\bar{n}} \sum_{\kappa^L \in \mathbb{K}} \sum_{\kappa^F \in \mathbb{K}} \varphi_{m,\kappa^L,\kappa^F} \left[\ln(\lambda^{\kappa^L}) x_{m,\kappa^L,\kappa^F} + \mathbb{1}_{\{\kappa^F > m\}} \ln(\lambda^{\kappa^F - m}) x_{-m,\kappa^F,\kappa^L} \right]. \end{aligned}$$

A.5 Analytical Results for a Cobb-Douglas Matching Function

Under a Cobb-Douglas function $\mathtt{M}(\mathtt{F},\mathtt{U}) = A \mathtt{F}^\gamma \mathtt{U}^{1-\gamma}$, job-finding and job-filling rates are $\mu(\theta) = \mathtt{M}(\theta, 1) = A\theta^\gamma$ and $\eta(\theta) = \mathtt{M}(1, \theta^{-1}) = A\theta^{\gamma-1}$, where $\theta \equiv \mathtt{F}/\mathtt{U}$. Using equation (18) at the BGP, the equilibrium market tightness is

$$\theta_\kappa(e) = \left(\frac{1}{A} \frac{\rho \boldsymbol{u}_\kappa - \boldsymbol{z}_\kappa}{e - \boldsymbol{u}_\kappa} \right)^{\frac{1}{\gamma}}. \tag{A.19}$$

The job-filling rate is:

$$\zeta_\kappa(e) \equiv \eta(\theta_\kappa(e)) = A^{\frac{1}{\gamma}} \left(\frac{e - \boldsymbol{u}_\kappa}{\rho \boldsymbol{u}_\kappa - \boldsymbol{z}_\kappa} \right)^{\frac{1-\gamma}{\gamma}}.$$

Hence, $\frac{\partial \zeta_\kappa(e)}{\partial e} = \frac{1-\gamma}{\gamma} \frac{\zeta_\kappa(e)}{e - \boldsymbol{u}_\kappa}$. Equation (25) then gives:

$$\left(\frac{1-\gamma}{\gamma} \right) \frac{\boldsymbol{\omega}(n,\kappa',\kappa^-) - \boldsymbol{\omega}(n,\kappa,\kappa^-) + \boldsymbol{u}_\kappa}{e'(n,\kappa',\kappa^-) - \boldsymbol{u}_{\kappa'}} = 1 + \left(\frac{1-\gamma}{\gamma} \right) \frac{e'(n,\kappa',\kappa^-)}{e'(n,\kappa',\kappa^-) - \boldsymbol{u}_{\kappa'}}$$

for any $(\kappa,\kappa',\kappa^-) \in \mathbb{K}^3$ and $n \in \{-\overline{n},\ldots,\overline{n}\}$. Solving for $e'(n,\kappa',\kappa^-)$, we then find:

$$e'_\kappa(n,\kappa',\kappa^-) = \gamma \boldsymbol{u}_{\kappa'} + (1-\gamma)\Big(\boldsymbol{\omega}(n,\kappa',\kappa^-) + \boldsymbol{u}_\kappa - \boldsymbol{\omega}(n,\kappa,\kappa^-) \Big). \tag{A.20}$$

The equilibrium job-filling rate for this firm in market segment κ' is then:

$$\zeta\Big(e'_\kappa(n,\kappa',\kappa^-)\Big) = \Big(A(1-\gamma)^{1-\gamma}\Big)^{\frac{1}{\gamma}} \left(\frac{\boldsymbol{\omega}(n,\kappa',\kappa^-) - \boldsymbol{\omega}(n,\kappa,\kappa^-) + \boldsymbol{u}_\kappa - \boldsymbol{u}_{\kappa'}}{\rho \boldsymbol{u}_{\kappa'} - \boldsymbol{z}_{\kappa'}} \right)^{\frac{1-\gamma}{\gamma}}. \tag{A.21}$$

Similarly, the continuation values promised by potential entrants are:

$$e^E_\kappa(m,\kappa^L,\kappa^F) = \gamma \boldsymbol{u}_\kappa + (1-\gamma)\boldsymbol{\omega}(-m,\kappa,\kappa^L), \quad \forall (m,\kappa^L,\kappa^F), \quad \forall m > 0,\ \forall(\kappa^L,\kappa^F) \tag{A.22}$$

$$e^E_\kappa(0,\kappa^L,\kappa^F) = \gamma \boldsymbol{u}_\kappa + (1-\gamma)\frac{1}{2}\Big(\boldsymbol{\omega}(0,\kappa,\kappa^L) + \boldsymbol{\omega}(0,\kappa,\kappa^F) \Big), \quad \forall(\kappa^L,\kappa^F), \tag{A.23}$$

so the entry rate in labor market segment κ and industry (m,κ^L,κ^F) is

$$\zeta_\kappa\Big(e^E_\kappa(m,\kappa^L,\kappa^F)\Big) = \Big(A(1-\gamma)^{1-\gamma}\Big)^{\frac{1}{\gamma}} \left(\frac{\boldsymbol{\omega}(-m,\kappa,\kappa^L) - \boldsymbol{u}_\kappa}{\rho \boldsymbol{u}_\kappa - \boldsymbol{z}_\kappa} \right)^{\frac{1-\gamma}{\gamma}}, \quad \forall(m,\kappa^L,\kappa^F), m > 0$$

$$\zeta_\kappa\Big(e^E_\kappa(0,\kappa^L,\kappa^F)\Big) = \Big(A(1-\gamma)^{1-\gamma}\Big)^{\frac{1}{\gamma}} \left(\frac{\frac{1}{2}\Big(\boldsymbol{\omega}(0,\kappa,\kappa^L) + \boldsymbol{\omega}(0,\kappa,\kappa^F) \Big) - \boldsymbol{u}_\kappa}{\rho \boldsymbol{u}_\kappa - \boldsymbol{z}_\kappa} \right)^{\frac{1-\gamma}{\gamma}}, \quad \forall(\kappa^L,\kappa^F).$$

Then, given optimal entry choices, the free entry condition in labor market $\kappa \in \{\underline{\kappa},\ldots,\overline{\kappa}\}$ reads:

$$c^e_\kappa = \gamma A^{\frac{1}{\gamma}} \left(\frac{1-\gamma}{\rho \boldsymbol{u}_\kappa - \boldsymbol{z}_\kappa} \right)^{\frac{1-\gamma}{\gamma}} \Bigg[\sum_{m=1}^{\overline{n}} \sum_{\kappa^L \in \mathbb{K}} \sum_{\kappa^F \in \mathbb{K}} \Big(\boldsymbol{\omega}(-m,\kappa,\kappa^L) - \boldsymbol{u}_\kappa \Big)^{\frac{1}{\gamma}} \tag{A.24}$$

$$+\sum_{\kappa^L\in\mathbb{K}}\sum_{\kappa^F\in\mathbb{K}}\left(\frac{1}{2}\Big(\boldsymbol{\omega}(0,\kappa,\kappa^L)+\boldsymbol{\omega}(0,\kappa,\kappa^F)\Big)-\boldsymbol{u}_\kappa\right)^{\frac{1}{\gamma}}\Bigg].$$

A.6 Computing the Elasticity of Innovation to R&D Spending

In our model, the total spending in R&D comes from innovation by the inventor and implementation by the firm. That is:

$$R(z,x)=\chi z^{\phi}+\xi x^{\phi}$$

Conversely, to achieve a spending R given an implementation rate x, the inventor needs to make effort:

$$z(R,x)=\left(\frac{R-\xi x^{\phi}}{\chi}\right)^{\frac{1}{\phi}}$$

From here, we can compute the elasticity of innovation to spending:

$$\frac{\partial z(R,x)}{\partial R}\frac{R}{z(R,x)}=\frac{1}{\phi}\left(1-\frac{\xi x^{\phi}}{R}\right)^{-1}$$

Substituting our expression for R:

$$\frac{\partial z(R,x)}{\partial R}\frac{R}{z(R,x)}=\frac{1}{\phi}\left(1-\frac{\xi x^{\phi}}{\chi z^{\phi}+\xi x^{\phi}}\right)^{-1}=\frac{1}{\phi}\left[1+\frac{\xi}{\chi}\left(\frac{x}{z}\right)^{\phi}\right]$$

For calibration, to pin down ϕ, we compute the average elasticity by using the equilibrium invariant distribution of industries as weights, and target the resulting number to be equal to 0.5.

A.7 Welfare

A.7.1 Initial Consumption, Output and Social Welfare

From equation (28) evaluated at $t=0$, initial consumption is given by:

$$\boldsymbol{C}_0=\boldsymbol{Y}_0\left(1-\int_0^1\xi(x_{ij0}^{\phi}+x_{-ij0}^{\phi})\,dj-\int_0^1\chi(z_{ij0}^{\phi}+z_{-ij0}^{\phi})\,dj-\sum_{\kappa\in\mathbb{K}}\left[\varphi_{\kappa 0}^{U}\left(\chi(z_{\kappa 0}^{U})^{\phi}\right)+c_{\kappa}^{e}\mathtt{m}^{P}\right]\right)\tag{A.25}$$

To compute initial output, in turn, we can use equation (A.15), so that:

$$\boldsymbol{Y}_0 = \exp\left[\int_0^1 \ln(q^L_{j0})dj + \int_0^1 \ln\left(s_{ij0}\left(\frac{M_{ij0}}{\boldsymbol{M}_0}\right)^{-1}\right)dj\right. \tag{A.26}$$
$$\left. + \frac{1}{\sigma}\int_0^1 \ln\left(1 + \left(\lambda^{-m_{j0}}\frac{s_{-ij0}}{s_{ij0}}\frac{M_{ij0}}{M_{-ij0}}\right)^{\sigma} + \left(\alpha\frac{s_{cj0}}{s_{ij0}}M_{ij0}\right)^{\sigma}\right)dj\right]$$

In a BGP, all terms in this equation are time-invariant, and we normalize average leader quality so that $\int_0^1 \ln(q^L_{j0})dj = 0$ across all counterfactual economies, which implies that the initial technological frontier is fixed (as the fringe keeps a fixed distance to the industry leader by assumption).

Along a BGP, we compute social welfare as the weighted sum of the lifetime utilities of the representative consumer, the attached inventors and the unattached inventors, so that

$$\begin{aligned}
\boldsymbol{W} &\equiv \int_0^{+\infty} e^{-\rho t}\left[\varpi^C\left(\underbrace{\ln\left(\frac{\boldsymbol{C}_t}{\boldsymbol{Y}_t}\right) + \ln(\boldsymbol{Y}_0) + gt}_{=\ln(\boldsymbol{C}_t)}\right) + \sum_{\kappa\in\mathbb{K}} \varpi^E_\kappa \boldsymbol{W}^E_\kappa \boldsymbol{Y}_0 e^{gt} + \sum_{\kappa\in\mathbb{K}} \varpi^U_\kappa \boldsymbol{u}_\kappa \boldsymbol{Y}_0 e^{gt}\right]dt \\
&= \varpi^C \frac{1}{\rho}\left[\ln\left(\frac{\boldsymbol{C}_0}{\boldsymbol{Y}_0}\right) + \ln(\boldsymbol{Y}_0) + \frac{g}{\rho}\right] + \frac{\boldsymbol{Y}_0}{\rho - g}\sum_{\kappa\in\mathbb{K}}\left(\varpi^E_\kappa \boldsymbol{W}^E_\kappa + \varpi^U_\kappa \boldsymbol{u}_\kappa\right)
\end{aligned}$$

where

$$\boldsymbol{W}^E_\kappa \equiv \sum_{m=0}^{\overline{n}} \sum_{\kappa^L=\underline{\kappa}}^{\overline{\kappa}} \sum_{\kappa^F\in\mathbb{K}} \varphi_{m,\kappa^L,\kappa^F}\left(e'_\kappa(m,\kappa^L,\kappa^F) + e'_\kappa(-m,\kappa^F,\kappa^L)\right)$$

where $e'_\kappa(\cdot)$ is the promised value defined in equation (A.20), and $\{\varpi^C, \{\varpi^E_\kappa\}_\kappa, \{\varpi^U_\kappa\}_\kappa\}$, with $\varpi^C + \sum_\kappa \varpi^E_\kappa + \sum_\kappa \varpi^U_\kappa = 1$, are the welfare weights on the representative consumer, the attached inventors and the unattached inventors, respectively.

A.7.2 Compensational Welfare

As this model features heterogeneous agents, there might be disagreements about policy. Therefore, we compute compensational welfare changes between two BGP equilibria (labeled A and B) for each of the three agent types separately.

For the consumer, we compute the percentage change in lifetime consumption that the representative consumer of economy "A" would require to remain indifferent between the two BGPs, or:

$$\frac{\varpi^{C,B}}{\rho}\left(\ln(\boldsymbol{C}_0^B)+\frac{g^B}{\rho}\right)=\frac{\varpi^{C,A}}{\rho}\left(\ln\left(\boldsymbol{C}_0^A(1+\iota^C)\right)+\frac{g^A}{\rho}\right)$$

Solving for ι^C:

$$\iota^C=\frac{1}{\boldsymbol{C}_0^A}\exp\left[\frac{\varpi^{C,B}}{\varpi^{C,A}}\left(\ln(\boldsymbol{C}_0^B)+\frac{g^B}{\rho}\right)-\frac{g^A}{\rho}\right]-1$$

For the attached inventor of type κ in economy A, the compensation ι_κ^E that makes her indifferent between the two BGPs solves:

$$\frac{\boldsymbol{Y}_0^B\varpi_\kappa^{E,B}\boldsymbol{W}_\kappa^{E,B}}{\rho-g^B}=\frac{\boldsymbol{Y}_0^A\varpi_\kappa^{E,A}\boldsymbol{W}_\kappa^{E,A}(1+\iota_\kappa^E)}{\rho-g^A}$$

Solving for ι_κ^E:

$$\iota_\kappa^E=\frac{\rho-g^A}{\rho-g^B}\left(\frac{\boldsymbol{Y}_0^B}{\boldsymbol{Y}_0^A}\frac{\varpi_\kappa^{E,B}}{\varpi_\kappa^{E,A}}\frac{\boldsymbol{W}_\kappa^{E,B}}{\boldsymbol{W}_\kappa^{E,A}}\right)-1$$

Finally, for the unattached inventor of type κ in economy A, the compensation ι_κ^U that makes her indifferent between the two BGPs solves:

$$\frac{\boldsymbol{Y}_0^B\varpi_\kappa^{U,B}\boldsymbol{u}_\kappa^{B}}{\rho-g^B}=\frac{\boldsymbol{Y}_0^A\varpi_\kappa^{U,A}\boldsymbol{u}_\kappa^{A}(1+\iota_\kappa^U)}{\rho-g^A}$$

Solving for ι_κ^U:

$$\iota_\kappa^U=\frac{\rho-g^A}{\rho-g^B}\left(\frac{\boldsymbol{Y}_0^B}{\boldsymbol{Y}_0^A}\frac{\varpi_\kappa^{U,B}}{\varpi_\kappa^{U,A}}\frac{\boldsymbol{u}_\kappa^{B}}{\boldsymbol{u}_\kappa^{A}}\right)-1$$

A.7.3 Choosing weights

The above welfare analysis requires choosing weights ϖ for the different agents. A natural candidate is to assume that these weights correspond to the population shares of each agent type:[42]

$$\varpi^C\equiv\frac{1}{3+\sum_{\kappa\in\mathbb{K}}\varphi_\kappa^U}\,,\qquad \varpi_\kappa^E\equiv\frac{\sum_n\sum_{\kappa^-}\varphi_{n,\kappa,\kappa^-}^E}{3+\sum_{\kappa\in\mathbb{K}}\varphi_\kappa^U}\,,\qquad \varpi_\kappa^U\equiv\frac{\varphi_\kappa^U}{3+\sum_{\kappa\in\mathbb{K}}\varphi_\kappa^U}$$

[42] The total measure of agents is $3+\sum_{\kappa\in\mathbb{K}}\varphi_\kappa^U$: a unit measure of consumers, a measure 2 of attached inventors, and an (endogenous) measure $\sum_{\kappa\in\mathbb{K}}\varphi_\kappa^U$ of unattached inventors.

where $\sum_n \sum_{\kappa^-} \varphi^E_{n,\kappa,\kappa^-}$ is the measure of attached inventors of type κ, with $\varphi^E_{n,\kappa,\kappa^-}$ defined in equation (A.12), and φ^U_κ is the measure of unattached inventors of type κ, i.e. the solution to equation (A.11).

In practice, however, this poses a conceptual complication. Consumers have log preferences whereas inventors have linear preferences, which generates a disagreement about how social transfers are evaluated by different agents. To account for these, one would have to compute compensational welfare across agents, i.e. inter-agent transfers that would make agents indifferent between being the consumer or an inventor at each point in time. This scheme of interpersonal transfers can certainly be complex to characterize, more so if one is to make inter-temporal comparisons between BGPs. Therefore, for simplicity, in practice we choose to focus only on the representative consumer by setting inventor weights to zero, i.e. $\varpi^C = 1$ and $\varpi^E_\kappa = \varpi^U_\kappa = 0$.

B Additional Empirical Results

This Appendix presents some additional results obtained from our merge sample of patent data (from PatentsView) and balance-sheet data (from Compustat).

B.1 Hiring policies within industries

First, we analyze how hiring and separation rates of inventors differ across firms with different sales within 6-digit NAICS industries. To do so, we estimate the following regression at the firm level:

$$Y_{it} = \beta_0 + \beta_1 \text{Market Share}_{it} + \beta_2 \text{Market Share}^2_{it} + \beta_3' X_{it} + \gamma_{s(i)t} + u_{it} \tag{B.1}$$

The outcome variables that we consider are: the firm hiring rate, separation rate and net hiring rate of inventors, defined as the difference between the first two. X_{it} includes controls for the firm's: age, employment, R&D stock, profitability, leverage and market-to-book ratio. $\gamma_{s(i)t}$ are industry-by-year fixed effects. Results are in Table B.1. We find that, within industries, there is an hump-shaped relationship between hiring rates and firms' relative sales and a U-shaped relationship between separation rates and firms' relative sales. As a result, we observe an hump-shaped relationship between a firm's *net* hiring rate of inventors and its market share.

We also compute hiring and separation rates separately for groups of inventors of different past productivity, identified by the quintiles of the inventors' cumulative productivity distribution in a given year. This allows us to characterize how hiring and separation patterns change for inventors of different past productivity across firms with different market share. As shown in Table B.2, the hump-shaped relationship between net hiring rates and firms' relative sales is preserved within each quintile of the inventor's productivity distribution, but tends to get more pronounced for the top quintiles. For a given market share, net hiring rates tend to be increasing in the inventor's productivity.

Table B.1: Hiring and separation rates of inventors and firm's relative sales

	(1) Hiring rate	(2) Separation rate	(3) Net hiring rate
Market Share	0.204***	-0.044***	0.248***
	(0.052)	(0.015)	(0.055)
Market Share^2	-0.197***	0.037**	-0.234***
	(0.059)	(0.017)	(0.063)
N	24,551	24,551	24,551
R^2	0.120	0.131	0.118
Controls	Yes	Yes	Yes
Year-industry FE	Yes	Yes	Yes

Notes: Standard errors clustered at firm and year level in parentheses.
* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Table B.2: Net hiring rates of inventors and firm's relative sales, by inventor productivity

	(1) Full sample	(2) Q1 Inv. Prod.	(3) Q2 Inv. Prod.	(4) Q3 Inv. Prod.	(5) Q4 Inv. Prod.	(6) Q5 Inv. Prod.
Market Share	0.204***	0.026***	0.053***	0.053***	0.067***	0.049**
	(0.052)	(0.009)	(0.013)	(0.015)	(0.017)	(0.017)
Market Share Squared	-0.197***	-0.023**	-0.051***	-0.055***	-0.063***	-0.043**
	(0.059)	(0.010)	(0.015)	(0.018)	(0.020)	(0.019)
N	24,551	24,551	24,551	24,551	24,551	24,551
R^2	0.120	0.109	0.104	0.120	0.099	0.085
Controls	Yes	Yes	Yes	Yes	Yes	Yes
Year-industry FE	Yes	Yes	Yes	Yes	Yes	Yes

Notes: Standard errors clustered at firm and year level in parentheses.
* $p<0.10$, ** $p<0.05$, *** $p<0.01$

B.2 Distribution of inventors across firms

We analyze the empirical patterns of inventor-firm sorting within industries by looking at how firms that differ in their relative sales differ in terms of the quality of the inventors that they employ. To do so, we compute the average productivity of the inventors attached to the firm in a given year, InvPvity_{it}, and estimate the following regression specification:

$$\log\left(\text{InvPvity}_{it}\right) = \beta_0 + \beta_1 \text{Market Share}_{it} + \beta_2 \text{Market Share}_{it}^2 + \boldsymbol{\beta}_3' \boldsymbol{X}_{it} + \gamma_{s(i)t} + u_{it} \quad \text{(B.2)}$$

where X_{it} includes the same controls as equation (B.1). The estimation results are reported in Table B.3 for alternative measures of inventors' productivity. Across these different measures, we see that there is a hump-shaped relationship between inventors' quality and firms' relative sales within industries. This tells us that, on average, firms with intermediate values of sales within the industry employ better inventors than firms that are far away from their competitors.

Table B.3: Inventor's productivity and firm's relative sales

	(1) Baseline	(2) 3-year window	(3) No self-cit.	(4) Cit. per author
Market Share	1.765***	1.914***	1.655***	1.577***
	(0.447)	(0.393)	(0.423)	(0.360)
Market Share^2	-1.766***	-1.727***	-1.681***	-1.522***
	(0.503)	(0.469)	(0.471)	(0.425)
N	15,858	13,691	15,654	19,174
R^2	0.142	0.183	0.136	0.120
Controls	Yes	Yes	Yes	Yes
Year-industry FE	Yes	Yes	Yes	Yes

Notes: Standard errors clustered at firm and year level in parentheses.
* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

B.3 Inventors' productivity and firms' innovation outcomes

To validate the model's predictions about inventors' productivity and firms' innovation outcomes, we look at this relationship in the data. We show that this relationship is heterogeneous across firms that face different degrees of competition in the product market. To do so, we estimate the following regression specification:

$$Y_{it} = \beta_0 + \beta_1 \log\left(\text{InvPvity}_{it}\right) + \boldsymbol{\beta}_2' \boldsymbol{X}_{it} + \gamma_{s(i)t} + u_{it} \quad \text{(B.3)}$$

We consider the following innovation outcomes: log R&D expenditure, number of patent applications, number of citations to these patents, average number of citations per patent application. X_{it} includes the same firm-level controls as above, plus the firm's market share and its square, and the number of inventors employed by the firm. $\gamma_{s(i)t}$ are industry-by-year fixed effects. The estimation results in Table B.4 show that firms that employ more productive inventors spend more in R&D, are granted more patents, have more total citations and average citations on their patents.

To study heterogeneity in the effects of inventors' productivity, we estimate equation (B.3) augmented with the interactions between Inventors Productivity$_{it}$, Market Share$_{it}$ and Market Share$^2_{it}$. The estimates, reported in Table B.5, show that employing more productive inventors is associated to improved innovation outcomes especially for firms with intermediate values of market share and less so for firms that are far from their competitors. This relationship holds and is statistically significant for the firm's R&D expenditure, patent applications and patent citations, while we do not detect any significant heterogeneity for the average number of citations per patent application.

Table B.4: Inventors' productivity and firms' innovation outcomes

	(1) R&D exp.	(2) Patents	(3) Citations	(4) Cit. per patent
Inventors' productivity	0.200***	0.624***	0.945***	1.209***
	(0.016)	(0.036)	(0.039)	(0.121)
Market Share	2.120***	2.359***	2.969***	1.198
	(0.653)	(0.843)	(0.895)	(1.178)
Market Share^2	-2.100***	-2.339**	-3.211***	-1.380
	(0.729)	(1.013)	(1.074)	(1.315)
N	14,104	15,858	15,828	15,858
R^2	0.794	0.882	0.876	0.049
Controls	Yes	Yes	Yes	Yes
Year-industry FE	Yes	Yes	Yes	Yes

Notes: Standard errors clustered at firm and year level in parentheses. Poisson regressions in (2), (3)
* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

B.4 Decomposing inventors' productivity

Several factors might affect the quantity and quality of an inventor's patents. To disentangle the contributions of inventor human capital and firm-specific innovation capabilities, we employ the Abowd *et al.* (1999) (AKM) statistical model. Based on our merged USPTO-Compustat sample, we construct a panel at the inventor level and estimate the following regression specification:

$$q_{jt} = \psi_i + \phi_{i(j,t)} + \gamma_t + X'_{jt}\beta + W'_{i(j,t)}\delta + u_{jt} \tag{B.4}$$

Table B.5: Inventors' productivity and firms' innovation outcomes, interactions with market share

	(1) R&D exp.	(2) Patents	(3) Citations	(4) Cit. per patent
Inventors' productivity	0.195***	0.579***	0.829***	1.229***
	(0.026)	(0.085)	(0.084)	(0.143)
Inventors' Prod. × Mkt Share	1.188**	2.424***	3.844***	-0.876
	(0.420)	(0.756)	(1.087)	(2.310)
Inventors' Prod. × Mkt Share^2	-2.356***	-4.474***	-6.115***	1.448
	(0.646)	(1.113)	(1.481)	(3.248)
N	14,106	15,862	15,832	15,862
R^2	0.566	0.820	0.816	0.049
Controls	Yes	Yes	Yes	Yes
Year-industry FE	Yes	Yes	Yes	Yes

Notes: Standard errors clustered at firm and year level in parentheses. Poisson regressions in (2), (3)
* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

where q_{jt} is our measure of inventor *flow* productivity defined in Section 2 as the quality of the patents that inventor j produced in year t. $\psi_i, \phi_{j(i,t)}, \gamma_t$ are inventor, firm and year fixed effects, respectively. X_{it} controls for inventor age and its square, and $W_{j(i,t)}$ includes the following firm-level controls: number of inventors at the firm, employment, age, R&D expenditure, leverage, profitability, market share and its square. We focus estimation on the largest set of firms connected by inventor moves, that is 82% of all firms in the sample.

Having estimated the model, we perform a variance decomposition exercise. We find that inventor fixed effects account for 87% of explained variance in q_{jt}, while firm fixed effects only account for 8% of the variance in inventor output explained by our model. Inventor and firm-level time-varying characteristics account for the remaining 5% of the explained variance in inventor productivity. Close to our results, using a similar methodology, Bhaskarabhatla, Cabral, Hegde and Peeters (2021) find that inventor heterogeneity accounts for 76% of explained variance in inventor productivity vs. 7% for firm heterogeneity. These results suggest inventor-specific human capital is the most important factor contributing to variation in inventor innovative output. This justifies using q_{jt}, and its cumulative version Q_{jt}, as a measure of inventor productivity.

C Additional Tables and Figures

Table C.1: Inventor productivity: Descriptive Statistics

Panel A: Patenting years

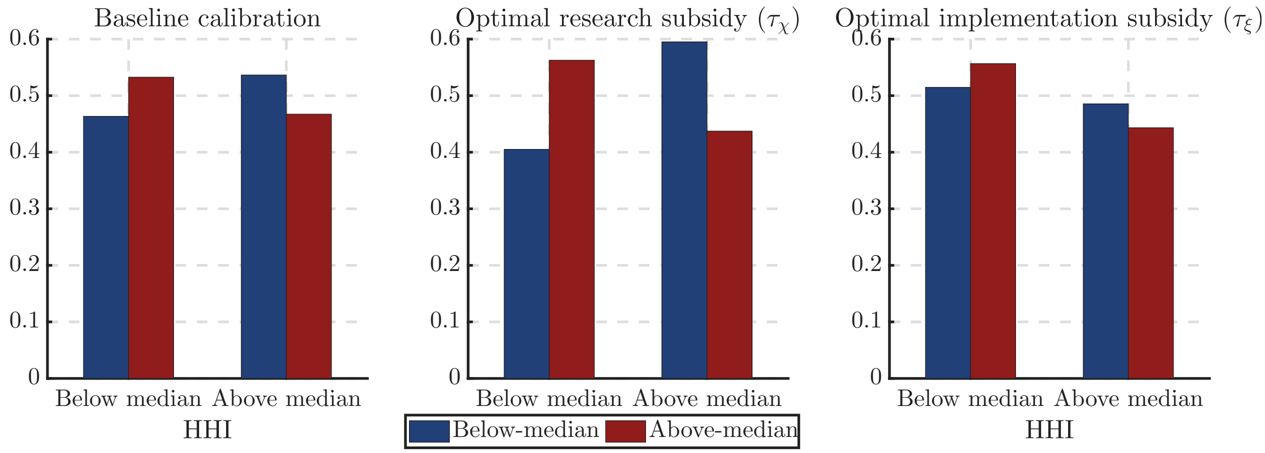
	Mean	Median	St. Dev.	Max	N
Patents per year	1.72	1	2.29	709	9,188,310
Total patents	4.98	2	13.6	5,850	3,180,835
Forward citations (5y) per year	4.66	1	25.5	20,176	9,188,310
Forward citations (3y) per year	2.02	1	14.3	10,273	9,188,310
Forward non-self-citations (5y) per year	10.4	3	32.7	10,618	6,138,010
Forward citations (5y) per author per year	1.76	0.5	8.1	5,790	9,188,310
Cumulative forward citations (5y)	3.77	1.44	10.9	4,864	6,007,475
Cumulative forward citations (3y)	1.64	0.615	5.23	2,972	6,007,475
Cumulative forward non-self-citations (5y)	5.86	2.22	13.9	2,497	4,132,809
Cumulative forward citations (5y) per author	1.43	.516	4.02	1,510	6,007,475

Panel B: Full inventor careers

	Mean	Median	St. Dev.	Max	N
Patents per year	0.8	0	1.78	709	19,790,453
Total patents	4.98	2	13.6	5,850	3,180,835
Forward citations (5y) per year	2.16	0	17.6	20,176	19,790,453
Forward citations (3y) per year	0.939	0	9.78	10,273	19,790,453
Forward non-self-citations (5y) per year	3.8	0	20.4	10,618	16,740,153
Forward citations (5y) per author per year	0.816	0	5.59	5,790	19,790,453
Cumulative forward citations (5y)	2.41	0.833	7.5	4,864	16,609,618
Cumulative forward citations (3y)	1.04	0.333	3.53	2,972	16,609,618
Cumulative forward non-self-citations (5y)	3.86	1	10.9	2,497	14,734,952
Cumulative forward citations (5y) per author	0.947	0.333	2.78	1,510	16,609,618

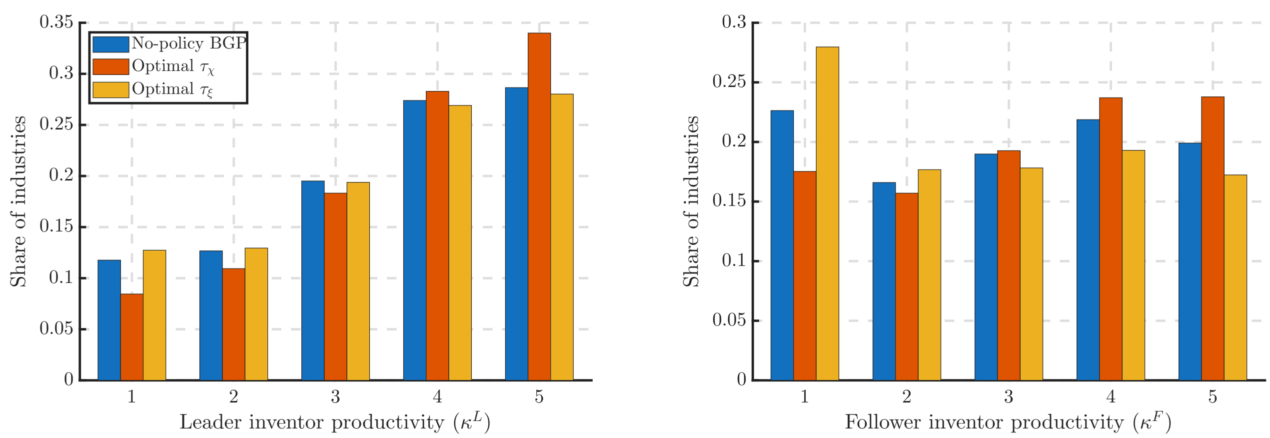
Notes: Descriptive statistics for various measures of inventor productivity, conditioning on inventors' patenting years (Panel A) and over the inventors' entire careers (Panel B).

Figure C.1: Effects of research an implementation subsidies on BGP equilibrium variables.



Notes: This figure reproduces Figure 7, for the baseline calibration (left), the optimal research subsidy (τ_χ), and the optimal implementation subsidy (τ_ξ).

Figure C.2: Equilibrium distribution of industries by inventor types attached to leader and follower firms.



Notes: This figure shows the distribution of industries by type of inventor attached to leader (left-hand panel) or follower (right-hand panel) firms, for the baseline calibration (blue bars), the optimal research subsidy τ_χ (orange bars), and the optimal implementation subsidy τ_ξ (yellow bars).